Lesson 13-6 From Number Puzzles to Properties of Integers

BIG IDEA Using algebra, you can show why divisibility tests and tricks relating to divisibility work.

In Lesson 2-3, you saw some number puzzles. In this lesson, you will see some unusual properties of divisibility that are like puzzles. Algebra shows why they work.

Activity 1

Step 1 Write down a 3-digit whole number, such as 175 or 220.

Step 2 Reverse the digits and subtract the new number from your original number.

$$\begin{array}{r} 175 \\ -\ 571 \\ \hline \end{array} \qquad \begin{array}{r} 220 \\ -\ 022 \\ \hline \end{array}$$

Step 3 Repeat Steps 1 and 2 with a few different numbers. You should find that the differences you get are always divisible by a large 2-digit number. What is that number?

Activity 2

1. Repeat Activity 1 with a few 4-digit numbers. Does the result you got in Activity 1 work for 4-digit numbers?
2. Does the result you got in Activity 1 work for 5-digit numbers?

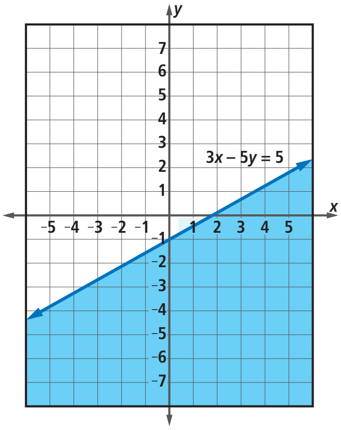
Activity 3

Step 1 Write down a 3-digit whole number.

Step 2 Next to your number from Step 1 write the same 3 digits, creating a 6-digit number.

Step 3 The 6-digit number you get is always divisible by 3 different small prime numbers. What are those numbers?

Mental Math



a. Describe the shaded region of the graph with an inequality.

b. Describe the unshaded region of the graph with an inequality.

Activity 4

Step 1 Write down spaces for the digits of an 8-digit number.

_?_ _?_, _?_ _?_ _?_, _?_ _?_ _?_

Step 2 Choose numbers for these digits so that the sum of the 1st, 3rd, 5th, and 7th digits equals the sum of the 2nd, 4th, 6th, and 8th digits.

Step 3 Try this with a few numbers. Find a 2-digit number less than 25 that divides the 8-digit number. You may want to use the FACTOR feature of a CAS.

Activity 5

Step 1 Create a 10-digit number using each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 once. For example, one such number is 8,627,053,914.

Step 2 Tell whether the statement is true or false.

a. Every number created in Step 1 will be divisible by 3.

b. Every number created in Step 1 will be divisible by 6.

c. Every number created in Step 1 will be divisible by 9.

d. Every number created in Step 1 will be divisible by 18.

e. Every number created in Step 1 will be divisible by 27.

Divisibility Properties Depending on the Rightmost Digits of Numbers

You have known for a long time that in the base-10 number system the 4-digit number 5,902 is a shorthand for $5 \cdot 1{,}000 + 9 \cdot 100 + 0 \cdot 10 + 2 \cdot 1$ or, using exponents, it is a shorthand for $5 \cdot 10^3 + 9 \cdot 10^2 + 0 \cdot 10^1 + 2 \cdot 10^0$.

We say that 2 is the units digit, or the digit in the units place, 0 is the tens digit, 9 is the hundreds digit, and 5 is the thousands digit for the number 5,902. In general, if u is the units digit, t is the tens digit, h is the hundreds digit, and T is the thousands digit, then the value of the 4-digit number is

$$1{,}000T + 100h + 10t + u.$$

You can extend this idea using more variables to give the value of any integer written in base 10 in terms of its digits.

By representing the value of a number in terms of its digits, you can prove some divisibility tests that you have known for a long time. The proofs are quite similar to those used in Lesson 13-5.

Example 1

Prove that if the units digit of a number in base 10 is even, then the number is even.

Solution The proof here is for a 4-digit number N. The proof for numbers with fewer or more digits is very similar. A 4-digit number in base 10 with digits as named above has the value

$$N = 1{,}000T + 100h + 10t + u.$$

If the units digit u is even, then $u = 2k$, where k is an integer. Substituting $2k$ for u, $N = 1{,}000T + 100h + 10t + 2k$.

Notice that 2 is a common monomial factor of the polynomial on the right side. Factor out the 2.

$$N = 2(500T + 50h + 5t + k)$$

Since $500T + 50h + 5t + k$ is an integer, N is twice an integer, so it must be even.

In a similar way, you can prove divisibility tests for 4, 5, 8, and 10.

Divisibility Tests Based on the Sum of the Digits

There is a different type of divisibility test for 9: just add the digits of the number. The number is divisible by 9 if and only if the sum of its digits is divisible by 9. Proving this involves a variation of the approach taken in Example 1.

Example 2

Prove that if the sum of the digits of a 4-digit integer written in base 10 is divisible by 9, then the number is divisible by 9.

Solution Call the number N. Suppose N has digits T, h, t, and u as named above. (The same idea holds for any number of digits.)

$$N = 1{,}000T + 100h + 10t + u$$

Now separate the sum of the digits from the value of the number.

$$N = (T + h + t + u) + (999T + 99h + 9t)$$

If the sum of the digits is divisible by 9, then there is an integer k with $T + h + t + u = 9k$. Substitute $9k$ for $T + h + t + u$.

$$N = 9k + (999T + 99h + 9t)$$
$$N = 9(k + 111T + 11h + t)$$

Since $k + 111T + 11h + t$ is an integer, N is divisible by 9.

GUIDED

Example 3

Prove that if the sum of the digits of a 4-digit integer written in base 10 is divisible by 3, then the number is divisible by 3.

Solution Use Example 2 as a model for your solution.

1. Call the number N. Suppose N has digits T, h, t, and u.

   $N = 1{,}000T + \underline{\ ?\ } + \underline{\ ?\ } + \underline{\ ?\ }$

2. Now separate the sum of the digits from the value of the number.

   $N = (T + h + t + u) + (\underline{\ ?\ } + \underline{\ ?\ } + \underline{\ ?\ })$

3. If the sum of the digits is divisible by 3 then there is an integer k with $T + h + t + u = 3k$.

4. Substitute into Step 2.

   $N = 3k + (\underline{\ ?\ } + \underline{\ ?\ } + \underline{\ ?\ })$

   $N = 3(\underline{\quad ?\quad })$

Because $\underline{\ ?\ }$ is an integer, N is divisible by 3.

Reversing Digits of a Number

Consider the 3-digit number $581 = 5 \cdot 100 + 8 \cdot 10 + 1$.

Reversing the digits of this number results in the number $185 = 1 \cdot 100 + 8 \cdot 10 + 5$.

So if a number has hundreds digit h, tens digit t, and units digit u, the number with the digits reversed has hundreds digit u, tens digit t, and units digit h. Whereas the first number has value $100h + 10t + u$, the number with its digits reversed has value $100u + 10t + h$.

Working with these numbers yields some surprising properties.

Example 4

Prove that if a 3-digit number is subtracted from the number formed by reversing its digits, then the difference is divisible by 99.

Solution Suppose the original number has the value $100h + 10t + u$.

Then the number with its digits reversed has value $100u + 10t + h$.

Subtracting the reversed number from the original yields the difference D.

$D = 100h + 10t + u - (100u + 10t + h)$
$D = 100h + 10t + u - 100u - 10t - h$
$D = 99h - 99u$
$D = 99(h - u)$

Since $h - u$ is an integer, the difference D is divisible by 99.

Questions

COVERING THE IDEAS

In 1–4, what is the value of the number?

1. The units digit of this 2-digit number is 7 and the tens digit is 5.
2. The units digit of this 2-digit number is u and the tens digit is t.
3. The thousands digit of the 4-digit number is A, the hundreds digit is B, the tens digit is C, and the units digit is D.
4. The millions digit of this 7-digit number is M, the thousands digit of this number is T, the units digit is 3, and all other digits are 0.
5. A 4-digit number has thousands digit T, hundreds digit h, tens digit t, and units digit u.
   a. What is the value of the number?
   b. What is the value of the number with its digits reversed?
6. Use Example 1 as a guide to prove: If the units digit of a 4-digit number in base 10 is 5, then the number is divisible by 5.
7. The proof in Example 2 is given for a 4-digit number. Adapt this proof for a 5-digit number, letting D be the ten-thousands digit.
8. **Fill in the Blanks** A number is divisible by 3 if and only if it can be written as _?_, where _?_ is an integer.

In 9–12, an integer is given.

a. Tell whether the integer is divisible by 2 and state a reason why.
b. Tell whether the integer is divisible by 5 and state a reason why.
c. Tell whether the integer is divisible by 9 and state a reason why.

9. 259,259,259
10. 225
11. 522
12. 522 − 225

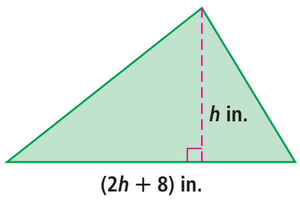
APPLYING THE MATHEMATICS

13. a. Find a counterexample: If a 4-digit number is subtracted from the number formed by reversing its digits, then the difference is divisible by 99.
    b. Prove: If a 4-digit number is subtracted from the number formed by reversing its digits, then the difference is divisible by 9.
14. a. Give an example of this statement and then prove it: If the units digit of a 5-digit number is 5 and the tens digit is 2, then the number is divisible by 25.
    b. Is the converse of the statement in Part a true?
15. In a certain 6-digit number, the hundred-thousands and hundreds digits are equal, the ten-thousands and tens digits are equal, and the thousands and units digits are equal. Prove that this number is divisible by 13.
16. The number $46x^3$, written in base 10, is divisible by 9. What is the value of x?
17. The tens digit of a 3-digit number is 4 times the hundreds digit and the number is divisible by 19. Find the number.

REVIEW

In 18 and 19, a statement is given. Prove the statement to show that it is true or provide a counterexample to show that it is false. (Lesson 13-5)

18. If a number is divisible by 5, then its square is divisible by 25.
19. If one number is divisible by 3, and a second number is divisible by 4, then the product of the two numbers is divisible by 7.
20. Give an example of an if-then statement that is false but whose converse is true. **(Lessons 13-2, 13-1)**
21. The triangle below has an area of 45 square inches. Find the height h of the triangle if the base is $2h + 8$ inches. **(Lesson 12-4)**



22. A rectangular box has dimensions a, $a + 3$, and $2a + 1$. **(Lessons 11-5, 11-2)**

a. Find a polynomial expression in standard form for the volume of the box.

b. What is the degree of the polynomial in Part a?

23. Consider the quadratic equation $4m^2 - 20m + 25 = 0$. **(Lesson 9-6)**

a. Find the value of the discriminant.

b. Use your answer to Part a to determine the number of real solutions to the equation.

In 24–26, solve the sentence. (Lessons 8-6, 5-9, 4-5)

24. $\sqrt{m - 10} = 3$ **25.** $5y - 2 > y$ **26.** $\frac{w + 27}{9} = \frac{w}{3}$

27. What is the value of x in the equation $\frac{(h^5)^{10} \cdot h^{15}}{h^{20}} = h^x$? **(Lessons 8-4, 8-3, 8-2)**

EXPLORATION

28. Let h, t, and u be the hundreds, tens, and units digits of a 3-digit number in base 10.

a. Find values of h, t, and u so that $hx^2 + tx + u$ is factorable over the integers.

b. For your values of h, t, and u, is it true that $100h + 10t + u$ is factorable over the integers?

c. True or False If h, t, and u are digits, and $hx^2 + tx + u$ is factorable over the set of polynomial with integer coefficients, then $100h + 10t + u$ is factorable over the integers.

d. Explore this statement to decide whether it is true or false: If h, t, and u are digits, and $100h + 10t + u$ is a prime number, then $hx^2 + tx + u$ is a prime polynomial over the integers.