Lesson

13-5 Proofs of Divisibility Properties

Vocabulary

closed under an operation
even integer, even number
odd integer, odd number
semiperimeter

BIG IDEA Using algebra, you can prove that even and odd numbers have certain general properties.

In this lesson, the following statements are assumed to be true.

The sum of two integers is an integer.

The difference of two integers is an integer.

The product of two integers is an integer.

The three properties above are examples of *closure properties.* A set is **closed under an operation** if the results of that operation always lies in that set. So another way of saying the above statements is:

The set of integers is closed under addition.

The set of integers is closed under subtraction.

The set of integers is closed under multiplication.

But the set of integers is *not* closed under division. For example, $8 \div 3$ is not an integer. When the quotient $a \div b$ is an integer, then we say that a is divisible by b, or that a is a multiple of b.

From the closure properties and other properties of real numbers that you know, it is possible to prove criteria that describe when one number is divisible by another.

Mental Math

A function contains $\{(-16, 4.5), (-7, 4.5), (0, 2), (1, -5), (5.5, 10)\}$.

a. State the domain.

b. State the range.

Divisibility by 2

An **even integer** (or **even number**) E is an integer that is twice another integer; that is, it is an integer that can be written as $2n$, where n is an integer. As you know, the integers are the numbers 0, 1, –1, 2, –2, 3, –3, ... So, if you multiply these numbers by 2, the results are the *even integers* 0, 2, –2, 4, –4, 6, –6, 8, –8, ...

Geometrically, a positive even number of dots can be arranged in two rows of the same length. For example, 14 dots can be split into two rows of 7.

To tell whether a large number is even, you cannot draw a pattern of dots. You must be able to show that it is twice another integer. For example, you can show that 5,734 is an even integer because $5{,}734 = 2 \cdot 2{,}867$, and 2,867 is an integer. How did we find 2,867? We divided 5,734 by 2. The number 0 is an even integer because $0 = 2 \cdot 0$. The negative number –88 is an even integer because $-88 = 2 \cdot -44$.

Odd Integers

An integer that is not even is called *odd.* Geometrically, an odd number of dots cannot be arranged in two rows of the same length. An example of this is shown with 15 dots. Notice below that there are two rows of 7 dots plus an additional dot.

You can see that an odd integer is one more than an even integer. So we define an **odd integer** (or **odd number**) as an integer that can be written as $2n + 1$, where n is an integer. For example, $2 \cdot (-54) + 1 = -107$, so –107 is odd.

STOP **QY**

If two numbers m and n are positive, you know that their sum $m + n$ and their product mn are positive. But the difference $m - n$ might be positive or negative. What happens if you know whether m and n are even or odd?

By trying some numbers, fill in the following table with one of the words "odd" or "even."

QY

Let $m = 87{,}654$ and $n = 3{,}210$. Tell whether these numbers are even or odd.

a. $m + n$

b. $m - n$

c. mn

Activity 1

m	n	$m + n$	$m - n$	mn
even	even	?	?	?
even	odd	?	?	?
odd	even	?	?	?
odd	odd	?	?	?

Testing pairs of numbers is not enough to show that a statement is true for *all* odd or even integers. Proofs are needed.

Example 1

Prove that the sum of two even numbers is an even number.

Solution To prove this statement, we think of it as an if-then statement: If two even numbers are added, then their sum is an even number.

Let m and n be the even numbers. Then, by the definition of even number, there are integers p and q with $m = 2p$ and $n = 2q$. So $m + n = 2p + 2q$.

By the Distributive Property of Multiplication over Addition, $m + n = 2(p + q)$.

Since the sum of two integers is an integer, $p + q$ is an integer.

So $m + n$, equal to 2 times an integer, is even.

Example 1 shows that the set of even numbers is closed under addition.

Using the idea of Example 1, you can prove that the difference of two even numbers is an even number. That is, the set of even numbers is closed under subtraction. (You are asked to write a proof of this in one of the questions at the end of the lesson.) Also, the product of two even numbers is an even number.

GUIDED

Example 2

Prove that the difference of two odd numbers is an even number.

Solution Let m and n be odd numbers. Then, by the definition of odd number, there are integers p and q with $m = 2p + 1$ and $n = 2q + 1$.

So $m - n = (\underline{\ ?\ }) - (\underline{\ ?\ }) = 2p - 2q$.

Thus $m - n = 2(\underline{\ ?\ })$.

Since the difference of integers is an integer, $\underline{\ ?\ }$ is an integer.

Consequently, $m - n$ is $\underline{\ ?\ }$ times an integer, so $m - n$ is even.

You can similarly prove that the sum of two odd numbers is an even number. The set of odd integers is *not* closed under addition or subtraction.

Example 3 deals with products.

Example 3

Prove that the product of two odd numbers is an odd number.

Solution Let m and n be odd numbers. Then, by the definition of odd number, there are integers p and q with $m = 2p + 1$ and $n = 2q + 1$.

Now multiply these numbers.

$mn = (2p + 1)(2q + 1)$

$= 4pq + 2p + 2q + 1$ Extended Distributive Property

$= 2(2pq + p + q) + 1$ Distributive Property (Common monomial factoring)

Since the product of two integers is an integer, 2pq is an integer. Since p and q are integers, the sum $2pq + p + q$ is an integer. This means that mn is 1 more than twice an integer, so mn is odd.

Divisibility by Other Numbers

A number is *divisible by 3* if and only if it can be written as $3n$, where n is an integer. Similarly, a number is *divisible by 4* if and only if it can be written as $4n$, where n is an integer. In general, a number is divisible by m if and only if it can be written as mn, where n is an integer.

Activity 2

Step 1 Let n be an odd positive integer. Fill in the table.

n	1	3	5	7	9	11	13	15	17
n^2	?	?	?	?	?	?	?	?	?
$n^2 - 1$	?	?	?	?	?	?	?	?	?

Step 2 **a.** What is the greatest common factor of the integers in the bottom row?

b. Is this number a factor of $n^2 - 1$ for all odd integers?

You cannot answer the last question in Activity 2 by just writing down more odd integers, squaring them, and subtracting 1. You can never show by examples that the statement is true for all odd integers. A proof is needed.

Example 4

Prove that the square of an odd integer is always 1 more than a multiple of 4.

Solution First find the square of an odd integer.

(continued on next page)

Let n be an odd integer. By definition of an odd integer, there is an integer k such that $n = 2k + 1$.

$$\begin{aligned} \text{So } n^2 &= (2k+1)^2 \\ &= (2k+1)(2k+1) \\ &= 4k^2 + 4k + 1 \\ n &= 4k(k+1) + 1 \end{aligned}$$

This shows that n^2 is 1 more than a multiple of 4.

Questions

COVERING THE IDEAS

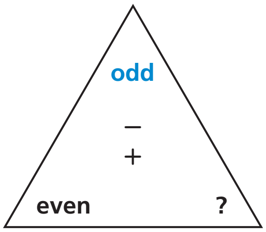
1. State the definition of *even integer.*
2. State the definition of *odd integer.*
3. Find a counterexample to show that this statement is not always true: *If two numbers are each divisible by 2, then their sum is divisible by 4.*

In 4 and 5, use Example 1 or Guided Example 2 as a guide to write a proof.

4. Prove: The difference of two even integers is an even integer.
5. **a.** Prove: The sum of two odd integers is an even integer.
   b. Is the set of odd integers closed under addition? Why or why not?
6. Use rows of dots to explain why the statement of Question 4 is true.
7. Use rows of dots to explain why the statement of Question 5 is true.
8. Prove that the set of even numbers is closed under multiplication. (*Hint:* Use Example 3 as a guide.)

APPLYING THE MATHEMATICS

In 9 and 10, complete the fact triangle. Then state the related facts.

9.


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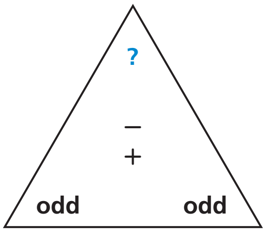


11. Prove: If a number is even, then its square is a multiple of 4.
12. Prove: If a number is divisible by 3, then its square is divisible by 9.

13. Prove: If the sum of two numbers is divisible by 35 and one of the two numbers is divisible by 70, then the other number is divisible by 35.

14. Prove or find a counterexample: If one number is divisible by 20 and a second number is divisible by 30, then their sum is divisible by 50.

15. Prove or find a counterexample: If one number is divisible by 4 and a second number is divisible by 6, then their product is divisible by 24.

REVIEW

16. Find two numbers whose sum is 562 and whose product is 74,865. **(Lesson 13-4)**

17. Find a value of b so that the quadratic expression $2x^2 - bx + 20$ is factorable over the integers. **(Lesson 12-6)**

In 18–23, solve the sentence. (Lessons 12-5, 9-5, 8-6, 4-5, 4-4, 3-4)

18. $100x^2 + 100x - 100 = 0$

19. $x^2 - 11x + 28 = 0$

20. $\frac{26}{N} = \frac{N}{0.5}$

21. $a \cdot 11^{\frac{1}{2}} = 99^{\frac{1}{2}}$

22. $4p - 12 \leq 60 - 5p$

23. $9.5 = 6x + 23.3$

24. Consider the system of equations $\begin{cases} 2x - 2y = 10 \\ -3x + 8y = -6 \end{cases}$. **(Lesson 10-8)**

a. Write the system in matrix form.

b. Use technology to find the inverse of the coefficient matrix.

c. Solve the system.

25. The **semiperimeter** of a triangle is half the perimeter of the triangle. Heron's formula (also called Hero's formula) shown below can be used to calculate the area A of a triangle given the lengths of the three sides a, b, and c. **(Lesson 8-6)**

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a+b+c)$$

a. If the side lengths of a triangle are 15, 9, and 12 inches, calculate the semiperimeter s of the triangle.

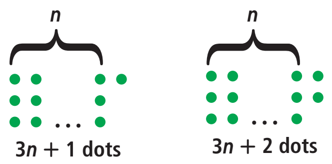
b. Find the area of the triangle in Part a.

Statue of Hero of Alexandria

26. If Emily reads 20 pages of a 418-page novel in 42 minutes, about how many hours will it take her to read the entire novel? **(Lesson 5-9)**

EXPLORATION

27. The numbers 1, 4, 7, 10, ..., which increase by 3, can be pictured as 3 equal rows of dots with 1 left over. These numbers are of the form $3n + 1$. The numbers 2, 5, 8, 11, ..., which increase by 3, can be pictured as 3 equal rows of dots with 2 left over. These numbers are of the form $3n + 2$.



What happens if you add, subtract, and multiply numbers of these forms? Are the answers all of the same form? Try to prove any results you find.

28. In Activity 2, you found that, for the first 9 odd positive integers, the square of the odd number is 1 more than a multiple of 8. Prove that the result is true for all odd integers.

QY ANSWERS

a. even

b. even

c. even