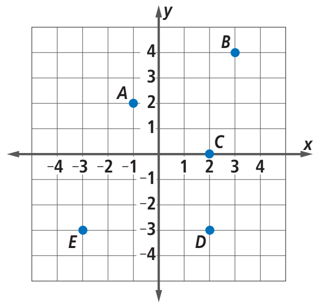
Lesson
13-4 A History and Proof of the Quadratic Formula

BIG IDEA The Quadratic Formula can be proved using the properties of numbers and operations.

The Quadratic Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is quite complicated. You may wonder how people used to solve quadratic equations before they had this formula, and how they discovered the Quadratic Formula in the first place. Here is some of the history.

Mental Math

Use the graph below to find each length.



a. *ED*

b. *CD*

c. *BC*

d. *AD*

What Problem First Led to Quadratics?

Our knowledge of ancient civilizations is based only on what survives today. The earliest known problems that led to quadratic equations are on Babylonian tablets dating from 1700 BCE. In these problems, the Babylonians were trying to find two numbers x and y that satisfy the system $\begin{cases} x + y = b \\ xy = c \end{cases}$.

This suggests that some Babylonians were interested in finding the dimensions x and y of a rectangle with a given area c and a given perimeter $2b$. The historian Victor Katz suggests that maybe there were some people who believed that if you knew the area of a rectangle, then you knew its perimeter. In solving these problems, these Babylonians may have been trying to show that many rectangles with different dimensions have the same area.

GUIDED

Example

Find the dimensions of a rectangular field whose perimeter is 300 meters and whose area is 4,400 square meters.

Solution Let *L* and *W* be the length and width of this rectangle.

Then $\begin{cases} \underline{\ ?\ } + \underline{\ ?\ } = 300 \\ \underline{\ ?\ } \cdot \underline{\ ?\ } = 4{,}400 \end{cases}$.

This system can be solved by substitution. First solve the top equation for *W*.

$\underline{\ ?\ } = 300 - \underline{\ ?\ }$

$W = 150 - \underline{\ ?\ }$

(continued on next page)

Now, substitute $\underline{\ ?\ }$ for W in the second equation.

$$L(\underline{\ ?\ }) = 4{,}400$$

This is a quadratic equation and so it can be solved by using either the Quadratic Formula or factoring to get $L = \underline{\ ?\ }$ or $L = \underline{\ ?\ }$. Now substitute these values for L in either of the original equations to get $W = \underline{\ ?\ }$ or $W = \underline{\ ?\ }$. So, the dimensions of the field are $\underline{\ ?\ }$ m by $\underline{\ ?\ }$ m.

How the Babylonians Solved Quadratics

The Babylonians, like the Greeks who came after them, used a geometric approach to solve problems like these. Using today's algebraic language and notation, here is what they did. It is a sneaky way to solve this sort of problem. Look back at the Example.

Because $L + W = 150$, the average of L and W is 75. This means that L is as much greater than 75 as W is less than 75. So let $L = 75 + x$ and $W = 75 - x$. Substitute these values into the second equation.

$$\begin{aligned} L \cdot W &= 4{,}400 \\ (75 + x)(75 - x) &= 4{,}400 \\ 5{,}625 - x^2 &= 4{,}400 \\ x^2 &= 1{,}225 \end{aligned}$$

Taking the square root, $x = 35$ or $x = -35$.

If $x = 35$:
$L = 75 + x$, so $L = 75 + 35 = 110$
$W = 75 - x$, so $W = 75 - 35 = 40$

If $x = -35$:
$L = 75 + -35 = 40$
$W = 75 - -35 = 110$

Either solution tells us that the field is 40 meters by 110 meters.

STOP **QY**

Notice what the Babylonians did. They took a complicated quadratic equation and, with a clever substitution, reduced it to an equation of the form $x^2 = k$. That equation is easy to solve. Then they substituted the solution back into the original equation.

This tablet contains 14 lines of a mathematical text in cuneiform script and a geometric design.

Source: Iraq Museum

QY

Use the Babylonian method to find two numbers whose sum is 72 and whose product is 1,007. (*Hint:* Let one of the numbers be $36 + x$, the other $36 - x$.)

The Work of Al-Khwarizmi

The work of the Babylonians was lost for many years. In 825 CE, about 2,500 years after the Babylonian tablets were created, a general method that is similar to today's Quadratic Formula was authored by the Arab mathematician Muhammad bin Musa al-Khwarizmi in a book titled *Hisab al-jabr w'al-muqabala*. Al-Khwarizmi's techniques were more general than those of the Babylonians. He gave a method to solve any equation of the form $ax^2 + bx = c$, where a, b, and c are positive numbers. His book was very influential. The word "al-jabr" in the title of his book led to our modern word "algebra." Our word "algorithm" comes from al-Khwarizmi's name.

Muhammad bin Musa al-Khwarizmi

A Proof of the Quadratic Formula

Neither the Babylonians nor al-Khwarizmi worked with an equation of the form $ax^2 + bx + c = 0$, because they considered only positive numbers, and if a, b, and c are positive, this equation has no positive solutions.

In 1545, a Renaissance scientist, Girolamo Cardano, blended al-Khwarizmi's solution with geometry to solve quadratic equations. He allowed negative solutions and even square roots of negative numbers that gave rise to complex numbers, a topic you will study in Advanced Algebra. In 1637, René Descartes published *La Géometrie* that contained the Quadratic Formula in the form we use today.

Now we prove why the formula works. Examine the argument in the following steps closely. See how each equation follows from the preceding equation. The idea is quite similar to the one used by the Babylonians, but a little more general. We work with the equation $ax^2 + bx + c = 0$ until the left side is a perfect square. Then the equation has the form $t^2 = k$, which you know how to solve for t.

Given the quadratic equation: $ax^2 + bx + c = 0$ with $a \neq 0$. We know $a \neq 0$ because otherwise the equation is not a quadratic equation.

Step 1 Multiply both sides of the equation by $\frac{1}{a}$. This makes the left term equal to $x^2 + \frac{b}{a}x + \frac{c}{a}$. The right side remains 0 because $0 \cdot \frac{1}{a} = 0$.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2 Add $-\frac{c}{a}$ to both sides in preparation for completing the square on the left side.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step 3 To complete the square add the square of half the coefficient of x to both sides. (See Lesson 12-2.)

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Step 4 The left side is now the square of a binomial.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Step 5 Take the power of the fraction to eliminate parentheses on the right side.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Step 6 To add the fractions on the right side, find a common denominator.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$$

Step 7 Add the fractions.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Step 8 Now the equation has the form $t^2 = k$, with $t = x + \frac{b}{2a}$ and $k = \frac{b^2 - 4ac}{4a^2}$. This is where the discriminant $b^2 - 4ac$ becomes important. If $b^2 - 4ac \geq 0$, then there are real solutions. They are found by taking the square roots of both sides.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Step 9 The square root of a quotient is the quotient of the square roots.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Step 10 This is beginning to look like the formula. Add $-\frac{b}{2a}$ to each side.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Step 11 Adding the fractions results in the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What if $b^2 - 4ac < 0$? Then the quadratic equation has no real number solutions. The formula still works, but you have to take square roots of negative numbers to get solutions. You will study these nonreal solutions in a later course.

Questions

COVERING THE IDEAS

1. **Multiple Choice** The earliest known problems that led to the solving of quadratic equations were studied about how many years ago?

   A 1,175 B 1,700
   C 2,500 D 3,700

2. In what civilization do quadratic equations first seem to have been considered and solved?

3. What is the significance of the work of al-Khwarizmi in the history of the Quadratic Formula?

In 4 and 5, suppose two numbers sum to 53 and have a product of 612. Show your work in finding the numbers.

4. Use the Quadratic Formula.

5. Use the Babylonian Method.

6. Suppose a rectangular room has a floor area of 54 square yards. Find two different lengths and widths that this floor might have.

In 7 and 8, suppose a rectangular room has a floor area of 144 square yards and that the perimeter of its floor is 50 yards.

7. Find its length and width by solving a quadratic equation using the Quadratic Formula or factoring.

8. Find its length and width using a more ancient method.

9. Find two numbers whose sum is 15 and whose product is 10.

10. In the proof of the Quadratic Formula, each of Steps 1–11 tells what was done but does not name the property of real numbers. For each step, name the property (or properties) from the following list.

    i. Addition Property of Equality
    ii. Multiplication Property of Equality
    iii. Distributive Property of Multiplication over Addition
    iv. Equal Fractions Property
    v. Power of a Quotient Property
    vi. Quotient of Square Roots Property
    vii. Definition of square root

APPLYING THE MATHEMATICS

11. Solve the equation $7x^2 - 6x - 1 = 0$ by following the steps in the derivation of the Quadratic Formula.

12. Explain why there are no real numbers x and y whose sum is 10 and whose product is 60.

13. In a Chinese text that is thousands of years old, the following problem is given: The height of a door is 6.8 more than its width. The distance between its corners is 10. Find the height and width of the door.

14. Here is an alternate proof of the Quadratic Formula. Tell what was done to get each step.

$$ax^2 + bx + c = 0$$

a. $4a^2x^2 + 4abx + 4ac = 0$

b. $4a^2x^2 + 4abx + 4ac + b^2 = b^2$

c. $4a^2x^2 + 4abx + b^2 = b^2 - 4ac$

d. $(2ax + b)^2 = b^2 - 4ac$

e. $2ax + b = \pm\sqrt{b^2 - 4ac}$

f. $2ax = -b \pm \sqrt{b^2 - 4ac}$

g. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

REVIEW

15. Consider the following statement. **(Lessons 13-2, 13-1)**

A number that is divisible by 8 is also divisible by 4.

a. Write the statement in if-then form.

b. Decide whether the statement you wrote in Part a is true or false. If it is false, find a counterexample.

c. Write the converse of the statement you wrote in Part a.

d. Decide whether the statement you wrote in Part c is true or false. If it is false, find a counterexample.

16. Solve $x^2 + 5x = 30$. **(Lesson 12-6)**

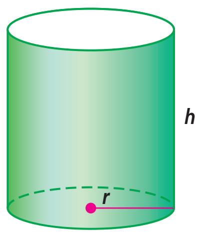
In 17–19, an open soup can has volume $V = \pi r^2 h$ and surface area $S = \pi r^2 + 2\pi rh$, where r is the radius and h is the height of the can.

17. Use common monomial factoring to rewrite the formula for S. **(Lesson 11-4)**

18. Find each of the following. **(Lesson 11-2)**

a. the degree of V

b. the degree of S



19. If the can has a diameter of 8 cm and a height of 12 cm, about how many milliliters of soup can it hold? (1 L = 1,000 cm^3) **(Lesson 5-4)**

20. Solve this system by graphing. $\begin{cases} y = |x| \\ y = \frac{1}{4}x^2 \end{cases}$ **(Lessons 10-1, 4-9)**

21. ***Skill Sequence*** Simplify each expression. **(Lessons 8-7, 8-6)**

a. $\sqrt{8} + \sqrt{5}$ **b.** $\sqrt{8} \cdot \sqrt{5}$ **c.** $\frac{\sqrt{8} \cdot \sqrt{5}}{\sqrt{2}}$

EXPLORATION

22. In a book or on the Internet, research al-Khwarizmi and find another contribution he made to mathematics or other sciences. Write a paragraph about your findings.

QY ANSWER

19 and 53