Lesson

13-3 Solving Equations as Proofs

Vocabulary

justifications
proof argument
deduction

▶ **BIG IDEA** Writing the steps in solving an equation and then checking the solutions together prove that you have found the only solutions to the equation.

Mental Math

Solve for *n*.

a. $5^n \cdot 5^4 = 5^{20}$

b. $(5^n)^4 = 5^{20}$

An Example of a Proof Argument

Every time you solve an equation, you are proving or deducing something true. Consider solving the equation $8x + 19 = 403$. The solution is 48. You may write the steps as we show here.

i. $8x + 19 = 403$
ii. $8x + 19 + -19 = 403 + -19$
iii. $8x + 0 = 384$
iv. $8x = 384$
v. $\frac{1}{8} \cdot 8x = \frac{1}{8} \cdot 384$
vi. $1 \cdot x = 48$
vii. $x = 48$

If you include the statements that show why each step follows from the previous steps, then you have written a *proof*. These statements are called **justifications.** They indicate why you can do what you have done to solve the equation.

	Conclusion	What Was Done	Justification
i.	$8x + 19 = 403$	Given	Given
ii.	$8x + 19 - 19 = 403 + -19$	Add –19 to both sides.	Addition Property of Equality
iii.	$8x + 0 = 384$	$19 + -19 = 0$	Additive Inverse Property
iv.	$8x = 384$	$8x + 0 = 8x$	Additive Identity Property
v.	$\frac{1}{8} \cdot 8x = \frac{1}{8} \cdot 384$	Multiply both sides by $\frac{1}{8}$.	Multiplication Property of Equality
vi.	$1x = 48$	$\frac{1}{8} \cdot 8 = 1$, $\frac{1}{8} \cdot 384 = 48$	Multiplicative Inverse Property
vii.	$x = 48$	$1 \cdot x = x$	Multiplicative Identity Property

With this argument, you have proved: *If* $8x + 19 = 403$*, then* $x = 48$. In this if-then statement, $8x + 19 = 403$ is the antecedent and $x = 48$ is the consequent. A **proof argument** in mathematics is a sequence of justified conclusions, starting with the antecedent and ending with the consequent. The use of a proof to show that one statement follows from another is called **deduction**.

A Justification Is Different from What Was Done

What you *do* to solve an equation is different from the justification. What was done applies to the specific equation being solved. The justification is the general property. In proofs, some people prefer to see what was done. Other people prefer the justification.

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Example 1

Prove if $-6x - 14 = 118$, then $x = -22$.

Solution

	Conclusion	What Was Done	Justification
i.	$-6x - 14 = 118$	Given	Given
ii.	$-6x - 14 + 14 = 118 + 14$	14 was added to both sides.	___?___
iii.	___?___	$-14 + 14 = 0$, $118 + 114 = 132$	Additive Inverse Property
iv.	$-6x = 132$	$-6x + 0 = -6x$	Arithmetic
v.	$-\frac{1}{6}(-6x) = -\frac{1}{6}(132)$	___?___	Multiplication Property of Equality
vi.	$1 \cdot x = -\frac{1}{6}(132)$	$-\frac{1}{6}(-6) = 1$	___?___
vii.	$x = -22$	___?___	___?___ Property

An Abbreviated Proof

Because work with additive and multiplicative inverses and identities is so automatic, some people prefer abbreviated proofs that do not show these steps. Here is an abbreviated proof of the statement: If $8x + 19 = 403$, then $x = 48$.

	Conclusion	Justification
i.	$8x + 19 = 403$	Given
ii.	$8x + 19 + -19 = 403 + -19$	Addition Property of Equality
iii.	$8x = 384$	Arithmetic
iv.	$\frac{1}{8} \cdot 8x = \frac{1}{8} \cdot 384$	Multiplication Property of Equality
v.	$x = 48$	Multiplicative Inverse Property

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Example 2

Prove that if $2x - 9 = 37$, then $x = 23$.

Solution Supply the justifications for the following conclusions.

	Conclusion	Justification
i.	$2x - 9 = 37$	?
ii.	$2x - 9 + 9 = 37 + 9$	?
iii.	$2x = 46$	?
iv.	$\frac{1}{2}(2x) = \frac{1}{2}(46)$	?
v.	$x = 23$	?

Justifications and Properties

In solving an equation or inequality, every justification is one of the following:

1. *Given* information (the given equation or inequality to be solved)
2. A property, of which there are three types:
   a. a *defined property,* such as the definition of slope, absolute value, or square root
   b. an *assumed property* of numbers, such as the Product of Powers Property or the Distributive Property
   c. a *previously-proved property,* such as the Means-Extremes Property or the Power of a Product Property, that were not assumed but proved using definitions or other known properties
3. *Arithmetic* (a catch-all term for all the properties you have learned that help you compute results of operations)

The Check Is a Converse

On the previous page you saw the proof of the statement: *If $8x + 19 = 403$, then $x = 48$.* When you check your work by substitution, you are proving: *If $x = 48$, then $8x + 19 = 403$.* The check is the converse of the solution. Together, the solution and the check mean *$8x + 19 = 403$ if and only if $x = 48$.*

Another way of saying this is: *$8x + 19 = 403$ exactly when $x = 48$.*

This means that 48 is a solution and no other numbers are solutions. Solving an equation means proving both a statement (to find the possible solutions) and its converse (to check that the possible solutions do work). Example 3 illustrates the importance of the check.

Example 3

Find all solutions to $\sqrt{x} = x - 6, x > 0$.

Solution

$\sqrt{x} = x - 6$	Given
$\sqrt{x} \cdot \sqrt{x} = (x - 6)(x - 6)$	Multiplication Property of Equality
$x = (x - 6)(x - 6)$	Definition of square root
$x = x^2 - 12x + 36$	Extended Distributive Property
$0 = x^2 - 13x + 36$	Addition Property of Equality
$x = \frac{13 \pm \sqrt{(-13)^2 - 4 \cdot 1 \cdot 36}}{2}$	Quadratic Formula
$x = \frac{13 \pm 5}{2}$	Simplify.
$x = 9$ or $x = 4$	Simplify.

This argument proves: If $\sqrt{x} = x - 6$, then $x = 9$ or $x = 4$. So, 9 and 4 are possible values of x. To see if they are solutions, a check is necessary.

Check When $x = 9$: Does $\sqrt{9} = 9 - 6$? Yes, $\sqrt{9} = 3$ and $9 - 6 = 3$. It checks.

When $x = 4$: Does $\sqrt{4} = 4 - 6$? No, $\sqrt{4} = 2$ and $4 - 6 = -2$. It does not check.

Consequently, $x = 9$ is the only solution to $\sqrt{x} = x - 6$. Putting it another way, $\sqrt{x} = x - 6$ if and only if $x = 9$.

In Example 3, the solution has proved: *If $\sqrt{x} = x - 6$, then $x = 9$ or $x = 4$.* The check has shown *If $x = 4$, then $\sqrt{x} \neq x - 6$ and if $x = 9$, then $\sqrt{x} = x - 6$.* So, *$\sqrt{x} = x - 6$ if and only if $x = 9$.*

Questions

COVERING THE IDEAS

1. Here is part of a proof argument. Explain what was done to get to Steps a–e and supply the missing justification.

$$40x + 12 = 3(6 + 13x)$$

a. $40x + 12 = 18 + 39x$

b. $40x + 12 + -12 = 18 + 39x + -12$

c. $40x + 0 = 18 + 39x + -12$

d. $40x = 18 + 39x + -12$

e. $40x = 6 + 39x$

f. The argument in Steps a–e proves what if-then statement?

2. Steps a–c are the conclusions in an abbreviated proof argument. Write what was done to get to each step and provide the justifications.

$$29 - 3y \le 44$$

a. $-29 + 29 - 3y \le -29 + 44$

b. $-3y \le 15$

c. $y \ge -5$

d. The argument in Steps a–c proves what if-then statement?

3. Explain the difference between inductive reasoning and deduction.

4. Give another example of inductive reasoning that is not written in this book.

5. After a month of timing her walks to school, Sula told her friend Lana, "It takes me 5 minutes less to get to school if I walk at a constant pace diagonally through the rectangular park than if I walk around two edges of its perimeter!" Lana replied, "That's a result of the Pythagorean Theorem!"

a. Who used inductive reasoning?

b. Who used deduction?

6. What five kinds of justifications are allowed in solving an equation or inequality?

7. a. Provide conclusions and justifications to prove that if $12m = 3m + 5$, then $m = \frac{5}{9}$.

b. What else do you need to do in order to prove that $12m = 3m + 5$ if and only if $m = \frac{5}{9}$.

8. Find all solutions to the equation $\sqrt{2n + 1} = n - 7$. You do not have to give justifications.

APPLYING THE MATHEMATICS

In 9–14, two consecutive steps of a proof are shown as an if-then statement. State what was done and state the justification.

9. If $d = rt$, then $r = \frac{d}{t}$.

10. If $3x + 5x = 80$, then $8x = 80$.

11. If $gn^2 + hn + k = 0$, then $n = \frac{-h \pm \sqrt{h^2 - 4gk}}{2g}$.

12. If $\sqrt{t} = 400$, then $t = 160{,}000$.

13. If $3x + 4y = 6$ and $3x - 4y = -18$, then $6x = -12$.

14. If $\frac{8}{3}b = 12$, then $8b = 36$.

15. Prove: If $ax + b = c$ and $a \neq 0$, then $x = \frac{c - b}{a}$. (*Hint:* Follow the steps of the solution to the first equation in the lesson.)

16. Show that if $x = \frac{c - b}{a}$, then $ax + b = c$.

17. Together, what do the statements in Questions 15 and 16 prove?

18. Use the definition of absolute value to find all values of x satisfying $|500x - 200| = 800$.

19. In Parts a and b, give abbreviated proofs.

a. Prove: If a and k are both positive and $a(x - h)^2 = k$, then $x = h \pm \sqrt{\frac{k}{a}}$.

b. Prove: If a and k are both positive and $x = h \pm \sqrt{\frac{k}{a}}$, then $a(x - h)^2 = k$.

c. What has been proved in Parts a and b?

20. Prove: The slope of the line with equation $Ax + By = C$ is the reciprocal of the slope of the line with equation $Bx + Ay = D$.

REVIEW

21. Consider the following statement to be true: Every person under 8 years of age receives a reduced fare on the metro city bus. **(Lessons 13-2, 13-1)**

a. Write this as an if-then statement.

b. Write the converse.

c. Is the converse true? Why or why not?

22. What value(s) can z not have in the expression $\frac{(2 - z)(1 + z)}{(4 + z)(3 - z)}$? **(Lessons 12-8, 5-2)**

23. **a.** Factor $3x^2 + 9x - 12$.

b. Find a value for x for which $3x^2 + 9x - 12$ is a prime number. **(Lesson 12-5)**

24. **a.** Solve the system $\begin{cases} 2y + 3x = 7 \\ y = 6x - 1 \end{cases}$.

b. Are the lines in Part a coincident, parallel, or intersecting? **(Lessons 10-6, 10-2)**

In 25–30, a property is stated. Describe the property using variables.

25. Product of Square Roots Property **(Lesson 8-7)**

26. Quotient of Square Roots Property **(Lesson 8-7)**

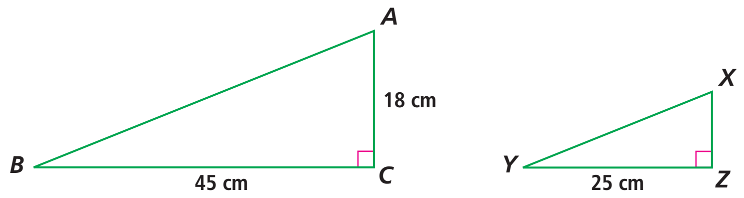
27. Power of a Power Property **(Lesson 8-2)**

28. Zero Product Property **(Lesson 2-8)**

29. Multiplication Property of –1 **(Lesson 2-4)**

30. Distributive Property of Multiplication over Subtraction **(Lesson 2-1)**

In 31 and 32, $\triangle ACB$ is similar to $\triangle XZY$. (Lesson 5-10, Previous Course)



31. Find the missing lengths.

32. **Fill in the Blank** If $m\angle Y \approx 22°$, then $m\angle B \approx$ __?__.

33. Two cards are drawn at random from a standard 52-card deck, without replacement. What is the probability of drawing an ace and a jack, in that order? **(Lesson 5-8)**

EXPLORATION

34. Use the Extended Distributive Property and other properties you have learned in this course to prove: $(a + b + c)(a + b - c)(b + c - a)(c + a - b) = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$.

The earliest playing cards are believed to have originated in Central Asia.

Source: The International Playing-Card Society