Lesson

6-10 Analyzing Solutions to Quadratic Equations

Vocabulary

discriminant

roots of an equation

zeros of a function

BIG IDEA You can determine whether the solutions to a quadratic equation with real coefficients are real or not real by calculating a value called the *discriminant* of the quadratic.

Mental Math

a. What size change is represented by the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$?

b. Give a matrix for $S_{4,2}$.

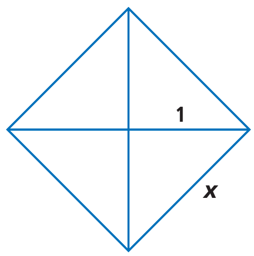
c. What scale change is represented by the matrix $\begin{bmatrix} 0.5 & 0 \\ 0 & 6 \end{bmatrix}$?

d. Give a matrix for $S_{7,2}$.

A Brief History of Quadratics

As early as 1700 BCE, ancient mathematicians considered problems that today would be solved using quadratic equations. The Babylonians described solutions to these problems using words that indicate they had general procedures for solving them similar to the Quadratic Formula. However, the ancients had neither our modern notation nor the notion of complex numbers. The history of the solving of quadratic equations helped lead to the acceptance of irrational numbers, negative numbers, and complex numbers.

The Pythagoreans in the 5th century BCE thought of x^2 as the area of a square with side x. So if $x^2 = 2$, as in the square pictured here, then $x = \sqrt{2}$. The Greeks proved that $\sqrt{2}$ was an irrational number, so a long time ago people realized that irrational numbers have meaning. But they never considered the negative solution to the equation $x^2 = 2$ because lengths could not be negative.



Writings of Indian and Arab mathematicians from 800 to 1200 CE indicate that they could solve quadratic equations. The Arab mathematician Al-Khowarizmi, in 825 CE in his book ***Hisab al-jabr w'al muqabala*** (from which we get the word "algebra"), solved quadratics like the Babylonians. His contribution is that he did not think of the unknown as having to stand for a length. Thus, the unknown became an abstract quantity. Around 1200, Al-Khowarizmi's book was translated into Latin by Fibonacci, and European mathematicians had a method for solving quadratics.

Mathematicians began using complex numbers in the 16th century because these numbers arose as solutions to quadratic and higher-degree equations. In the 19th century, Gauss brought both geometric and physical meaning to complex numbers. The geometric meaning built on Descartes' coordinate plane. Physical meanings of complex numbers occur in a variety of engineering and physics applications. In 1848, Gauss was the first to allow the *coefficients* in his equations to be complex numbers. Today, complex numbers are used in virtually all areas of mathematics.

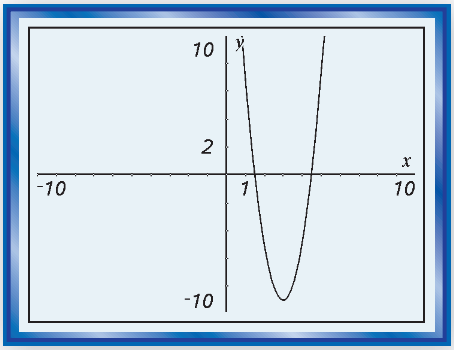
Predicting the Number of Real Solutions to a Quadratic Equation

Activity

Work in groups of three. Record all the results for Steps 1, 2 and 4 in a single table like the one below. For Steps 1–4, divide the six equations in the table equally among the group members.

$y = ax^2 + bx + c$	Number of x-intercepts of Graph	Solutions to $ax^2 + bx + c = 0$	Number of Real Solutions to $ax^2 + bx + c = 0$	Value of $b^2 - 4ac$
a. $y = 4x^2 - 24x + 27$	?	?	?	?
b. $y = 4x^2 - 24x + 36$	?	?	?	?
c. $y = 4x^2 - 24x + 45$	?	?	?	?
d. $y = -6x^2 + 36x - 54$	?	?	?	?
e. $y = -6x^2 + 36x - 48$	?	?	?	?
f. $y = -6x^2 + 36x - 60$	?	?	?	?

Step 1 Graph each quadratic equation and record the number of x-intercepts of its graph.



Step 2 Solve each equation using the Quadratic Formula and record how many of the solutions are real.

Step 3 Describe any patterns you see in the table so far.

Step 4 When $ax^2 + bx + c = 0$, the value of the expression $b^2 - 4ac$ can be used to predict the number of real solutions to the quadratic equation. This value is the **discriminant** of the quadratic equation. Calculate and record the discriminant for each equation in the table. Make a conjecture about the relationship between the number of real solutions and the value of the discriminant.

(continued on next page)

Step 5 **a.** Have each person make three new quadratic equations of the form $y = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. One equation should have two real solutions, one should have one real solution, and the third should have no real solutions.

b. Add each of the equations in Part a to the table, then graph them on your graphing utility. Solve each equation, then record the number of x-intercepts, the number of real solutions, the values of the real solutions, and the value of the discriminant in the table.

c. Do the nine additions to the table support your conjecture from Step 4?

How Many Real Solutions Does a Quadratic Equation Have?

Now consider the general quadratic equation. When the coefficients a, b, and c are real numbers and $a \neq 0$, the Quadratic Formula gives the following two solutions to $ax^2 + bx + c = 0$: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

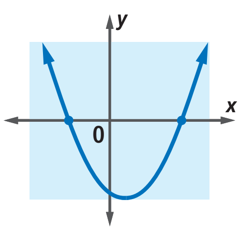
STOP QY

▶ QY

Write the two solutions to the Quadratic Formula as a compound sentence.

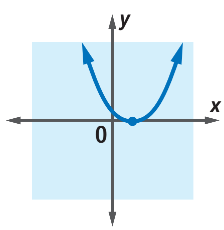
Because a and b are real numbers, the numbers $-b$ and $2a$ are real, so only $\sqrt{b^2 - 4ac}$ could possibly not be real. It is because of this property that the number $b^2 - 4ac$ is called the discriminant. It allows you to discriminate the *nature of the solutions* to the equation, as shown below.

If $b^2 - 4ac$ is positive, then $\sqrt{b^2 - 4ac}$ is a positive number. There are two real solutions. The graph of $y = ax^2 + bx + c$ intersects the x-axis in two points.



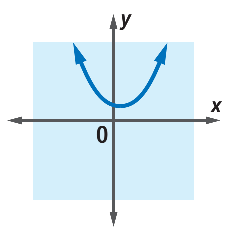
$y = ax^2 + bx + c$
$b^2 - 4ac > 0$
two real solutions

If $b^2 - 4ac$ is zero, then $\sqrt{b^2 - 4ac} = \sqrt{0} = 0$. Then $x = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$, and there is only one real solution. The graph of $y = ax^2 + bx + c$ intersects the x-axis in one point.



$y = ax^2 + bx + c$
$b^2 - 4ac = 0$
one real solution

If $b^2 - 4ac$ is negative, then $\sqrt{b^2 - 4ac}$ is an imaginary number. There will then be two nonreal solutions. Furthermore, because these solutions are of the form $m + ni$ and $m - ni$, they are complex conjugates. The graph of $y = ax^2 + bx + c$ does not intersect the x-axis.



$y = ax^2 + bx + c$
$b^2 - 4ac < 0$
two nonreal solutions

The graphs on the previous page are drawn for positive a, so the parabolas open up. Solutions to quadratic (and other) equations are also called **roots of the equation** $ax^2 + bx + c = 0$, or **zeros of the function** represented by the equation $y = ax^2 + bx + c$. The number i allows you to write square roots of negative numbers as complex solutions. You saw in the Activity that the number of real roots of each quadratic equation equals the number of x-intercepts. To summarize, the results of the Activity should be consistent with the following theorem.

Discriminant Theorem

Suppose a, b, and c are real numbers with $a \neq 0$.
Then the equation $ax^2 + bx + c = 0$ has:

(i) two real solutions, if $b^2 - 4ac > 0$.

(ii) one real solution, if $b^2 - 4ac = 0$.

(iii) two complex conjugate solutions, if $b^2 - 4ac < 0$.

GUIDED

Example 1

Determine the nature of the roots of each equation.

a. $4x^2 - 4x + 1 = 0$

b. $25x^2 + 6x + 4 = 0$

c. $3x^2 + 5x - 14 = 0$

Solution Use the Discriminant Theorem. Let $D = b^2 - 4ac$.

a. Here $a = 4$, $b = -4$, and $c = 1$. So $D = (-4)^2 - 4(4)(1) = 0$. Thus, the equation has __?__ real root.

b. Here $a = 25$, $b = 6$, and $c = 4$, so $D =$ __?__. Because D is negative, the equation has __?__ real roots.

c. Here $a =$ __?__, $b =$ __?__, and $c =$ __?__; so $D =$ __?__ > 0. So, this equation has __?__ real roots. Because D is not a perfect square, the roots are irrational.

Check Use a graphing utility. Let $f1(x) = 4x^2 - 4x + 1$, $f2(x) = 25x^2 + 6x + 4$, and $f3(x) = 3x^2 + 5x - 14$. The number of real solutions should equal the number of x-intercepts of the graph.

a. The graph of $f1$ has __?__ x-intercept. So, the equation has __?__ real root. It checks.

b. The graph of $f2$ has __?__ x-intercepts. So, the equation has __?__ real roots. It checks.

c. The graph of $f3$ has __?__ x-intercepts. So, the equation has __?__ real roots. It checks.

Applying the Discriminant Theorem

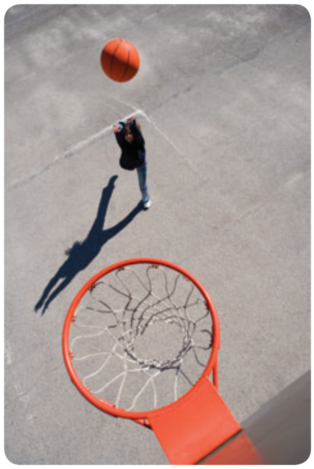
The number of real solutions to a quadratic equation can tell you something about the situation that led to the equation. The following example shows how.

Example 2

Eight-year old Allie Oop, an aspiring basketball player, shoots five feet from the hoop. She is trying to get the ball above the rim, which is set at a height of 10 feet. She releases the ball from an initial height of 4 feet. The following equation models the path of the ball, where *x* is the horizontal distance in feet that the ball has traveled and *h* is the ball's height in feet above the ground.

$$h = -0.4x^2 + 3x + 4$$

Use the Discriminant Theorem to determine whether Allie's ball ever reaches the height of the rim.



Solution The ball will reach the rim if there are real values of x for which

$$10 = -0.4x^2 + 3x + 4.$$

First, rewrite the equation in standard form. $0 = -0.4x^2 + 3x - 6$

Then calculate the value of the discriminant. $3^2 - 4(-0.4)(-6) = -0.6$

The discriminant $D = -0.6$. Because D is negative, there are no real solutions to this equation. This means that the ball will not reach the height of the rim.

Questions

COVERING THE IDEAS

1. What is the relationship between the number of x-intercepts of the graph of a quadratic function and the number of real solutions to the corresponding quadratic equation?
2. Why did the Pythagoreans think there was only one solution to $x^2 = 2$?
3. Consider the equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers.
   a. What is its discriminant?
   b. What are its roots?
4. The discriminant of an equation $ax^2 + bx + c = 0$ is 0. What does this indicate about the graph of $y = ax^2 + bx + c$?

5. **Matching** Match the idea about quadratics at the left with the estimated length of time that idea has been understood.

a. geometric and physical meanings to complex numbers	i. about 3700 years
b. problems in which the unknown could be an abstract quantity	ii. about 1400 years
	iii. about 1750 years
	iv. about 1150 years
	v. about 150 years

In 6 and 7, a quadratic expression is given.

a. Set the expression equal to 0. Use the discriminant to determine the nature of the roots to the equation.

b. Set the expression equal to *y*. How many *x*-intercepts does the graph of the equation have?

6. $2x^2 - 3x + 7$

7. $x^2 - 10x + 9$

8. a. Solve $2s^2 - 3s + 7 = 0$, and write the solutions in $a + bi$ form.

   b. **True or False** The roots of this equation are complex conjugates.

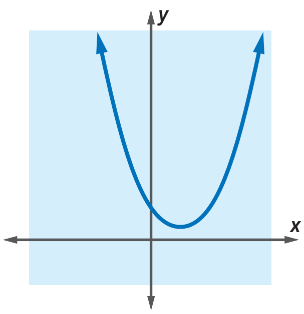
9. Sketch a graph of a quadratic function $y = ax^2 + bx + c$, with $a > 0$ and a negative discriminant similar to one of the three graphs on page 436.

In 10 and 11, a graph of a quadratic function $f: x \longrightarrow ax^2 + bx + c$ is given.

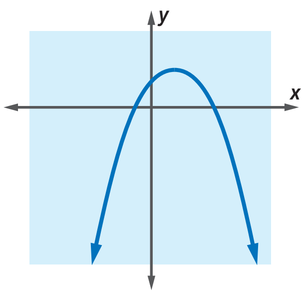
a. Tell whether the value of $b^2 - 4ac$ is positive, negative, or zero.

b. Tell how many real roots the equation $f(x) = 0$ has.

10.



11.



12. Without drawing a graph, tell whether the x-intercepts of $y = 2.5x^2 - 10x + 6.4$ are rational or irrational. Justify your answer.

13. Refer to Example 2. Allie practices her shot, but it never reaches the rim, so her father lowers the hoop to 9 feet. Now will the ball reach the height of the rim? Justify your answer.

APPLYING THE MATHEMATICS

In 14 and 15, True or False. If true, explain why; if false, give a counterexample.

14. The y-intercept of every parabola that has an equation of the form $y = ax^2 + bx + c$ is $(0, b)$.
15. Whenever a parabola opens down and its vertex is above the x-axis, its discriminant must be positive.
16. a. Find the value(s) of k for which the graph of the equation $y = x^2 + kx + 6.25$ will have exactly one x-intercept.
    b. Check your answer to Part a by graphing.
17. Can a quadratic equation have two real roots, one rational and the other irrational? Why or why not?

REVIEW

18. Write $\frac{49+28i}{2+3i}$ in $a + bi$ form by hand. **(Lesson 6-9)**
19. Solve the equation $x^2 - 8x + 25 = 0$. Write the solutions in $a + bi$ form. **(Lesson 6-9)**
20. Find all the *fourth* roots of 16, that is, the square roots of the square roots of 16. **(Lesson 6-8)**
21. Penny drops a penny from the top of a 1200-foot building. At the exact moment Penny drops the penny, her sister Ellie gets on an elevator at a height of 300 feet which travels upward at a constant rate of 30 $\frac{\text{feet}}{\text{second}}$. Ignore air resistance. **(Lessons 6-4, 5-2, 3-1)**
    a. Write an equation for the height p in feet of Penny's penny as a function of the time t in seconds after it is dropped.
    b. Write an equation for Ellie's height m as a function of t.
    c. Graph your two equations from Parts a and b and use the graph to determine at what time are Ellie and the penny at the same height above the ground.
22. The trim around a square window is 2 inches wide. The total length of one side of the window including trim is x inches. Find an expression in standard form for the area of just the window without trim. **(Lesson 6-1)**

EXPLORATION

23. The word "algebra" comes from the Arabic "al-jabr." What does "al-jabr" mean and what does that meaning have to do with algebra?

QY ANSWER

$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or

$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$