Lesson 6-9 Complex Numbers

Vocabulary

complex number
real part, imaginary part
equal complex numbers
complex conjugate

▶ **BIG IDEA** *Complex numbers* are numbers of the form $a + bi$, where $i = \sqrt{-1}$, and are operated with as if they are polynomials in i.

Mental Math

Write an inequality to represent the sentence.

a. The weight w of my carry-on luggage must be less than 30 pounds.

b. A medium-size sock can be worn by anyone with a shoe size s from 7 to 10.

c. To ride this roller coaster, your height h must be at least 54 inches.

Many aspects of an electrical charge, such as voltage (electric potential) and current (movement of an electric charge), affect the performance and safety of the charge. When working with these two quantities, electricians find it easier to combine them into one number Z, called *impedance*. Impedance in an alternating-current (AC) circuit is the amount, usually measured in ohms, by which the circuit resists the flow of electricity. The two-part number Z is the sum of a real number and an imaginary number, and is called a *complex number*.

What Are Complex Numbers?

Recall from the previous lesson that the set of numbers of the form bi, where b is a real number, are called *pure imaginary numbers*. When a real number and a pure imaginary number are added, the sum is called a *complex number*.

Definition of Complex Number

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

In the complex number $a + bi$, a is the **real part** and b is the **imaginary part**. For example, $-8.5 - 4i$ is a complex number in which the real part is -8.5 and the imaginary part is -4 (not $4i$ or $-4i$).

We say that $a + bi$ and $c + di$ are **equal complex numbers** if and only if their real parts are equal and their imaginary parts are equal. That is, $a + bi = c + di$ if and only if $a = c$ and $b = d$. For example, if $x + yi = 2i - 3$, then $x = -3$ and $y = 2$.

STOP QY1

▶ QY1

If $a + bi = -4 + \sqrt{-3}$, what is a and what is b?

Operations with Complex Numbers

All of the assumed properties of addition, subtraction, multiplication, and division of real numbers hold for complex numbers.

Properties of Complex Numbers Postulate

In the set of complex numbers:

1. Addition and multiplication are commutative and associative.
2. Multiplication distributes over addition and subtraction.
3. $0 = 0i = 0 + 0i$ is the additive identity; $1 = 1 + 0i$ is the multiplicative identity.
4. Every complex number $a + bi$ has an additive inverse $-a + -bi$ and a multiplicative inverse $\frac{1}{a + bi}$ provided $a + bi \neq 0$.
5. The addition and multiplication properties of equality hold.

You can use the properties to operate with complex numbers in a manner consistent with the way you operate with real numbers. You can also operate with complex numbers on a CAS.

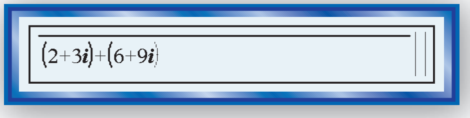
Activity

Step 1 Add the complex numbers.

$(2 + 3i) + (6 + 9i) =$ ___?___

$(4 - 3i) + (7 + 5i) =$ ___?___

$(-16 + 5i) + (4 - 8i) =$ ___?___



Step 2 Subtract the complex numbers.

$(4 - 3i) - (6 + 5i) =$ ___?___

$(-2 + i) - (7 + 9i) =$ ___?___

$(8 - 4i) - (1 - i) =$ ___?___

Step 3 Describe, in words and using algebra, how to add and subtract two complex numbers.

Step 4 Check your answer to Step 3 by calculating $(a + bi) + (c + di)$ and $(a + bi) - (c + di)$ on a CAS.

In the Activity, you should have seen that the sum or difference of two complex numbers is a complex number whose real part is the sum or difference of the real parts and whose imaginary part is the sum or difference of the imaginary parts.

The Distributive Property can also be used to multiply a complex number by a real number or by a pure imaginary number.

Example 1

Put $4i(8 + 5i)$ in $a + bi$ form.

Solution

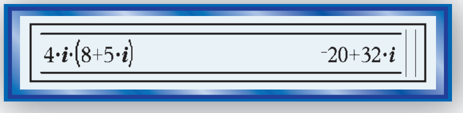
$4i(8 + 5i) = 4i(8) + 4i(5i)$ Distributive Property

$= 32i + 20(i^2)$ Associative and Commutative Properties of Multiplication

$= 32i + 20(-1)$ Definition of i

$= -20 + 32i$ Commutative Property of Addition

Check Multiply on a CAS. It checks.



In Example 1, notice that i^2 was simplified using the fact that $i^2 = -1$. Generally, you should write answers to complex number operations in $a + bi$ form. Most calculators use this form as well.

To multiply complex numbers, think of them as linear expressions in i and multiply using the Distributive Property. Then use $i^2 = -1$ to simplify your answer.

Example 2

Multiply and simplify $(6 - 2i)(4 + 3i)$.

Solution

$(6 - 2i)(4 + 3i) = 24 + 18i - 8i - 6i^2$ Distributive Property (Expand.)

$= 24 + 10i - 6i^2$ Distributive Property (Combine like terms.)

$= 24 + 10i - 6(-1)$ Definition of i

$= 30 + 10i$ Arithmetic

STOP QY2

QY2

Multiply $(4 + 3i)(4 - 3i)$.

Conjugate Complex Numbers

The complex numbers $4 + 3i$ and $4 - 3i$ in QY 2 are *complex conjugates* of each other. In general, the **complex conjugate** of $a + bi$ is $a - bi$. Notice that the product $(4 + 3i)(4 - 3i)$ is a real number. In Question 23, you are asked to prove that the product of any two complex conjugates is a real number.

Complex conjugates are useful when dividing complex numbers. To divide two complex numbers, multiply both numerator and denominator by the conjugate of the denominator. This gives a real number in the denominator that you can then divide into each part of the numerator.

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Example 3

Simplify $\frac{3+6i}{3-2i}$.

Solution Multiply the numerator and denominator by $3 + 2i$, the conjugate of $3 - 2i$.

$\frac{3+6i}{3-2i} = \frac{3+6i}{3-2i} \bullet \frac{3+2i}{3+2i}$ Identity Property of Multiplication

$= \frac{(3+6i)(3+2i)}{(3-2i)(3+2i)}$ Multiplication of fractions

$= \frac{?}{?}$ Distributive Property (Expand.)

$= \frac{?}{9-4i^2}$ Distributive Property (Combine like terms.)

$= \frac{?}{9-4(?)}$ Definition of i

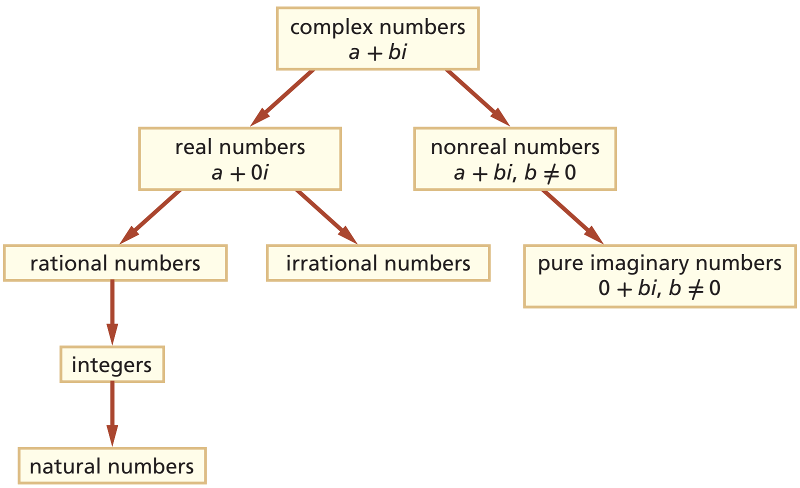
$= \frac{?}{13} + \frac{?}{13}i$ Distributive Property (adding fractions)

Check Divide on a CAS. It checks.

The Various Kinds of Complex Numbers

Because $a + 0i = a$, every real number a is a complex number. Thus, the set of real numbers is a subset of the set of complex numbers. Likewise, every pure imaginary number bi equals $0 + bi$, so the set of pure imaginary numbers is also a subset of the set of complex numbers.

The diagram at the right is a *hierarchy of number sets*. It shows how the set of complex numbers includes some other number sets.



Applications of Complex Numbers

The first use of the term *complex number* is generally credited to Carl Friedrich Gauss. Gauss applied complex numbers to the study of electricity. Later in the 19th century, applications using complex numbers were found in geometry and acoustics. In the 1970s, complex numbers were used in a new field called *dynamical systems*.

Recall that electrical impedance Z is defined as a complex number involving voltage V and current I. A complex number representing impedance is of the form $Z = V + Ii$.

The total impedance Z_T of a circuit made from two connected circuits is a function of the impedances Z_1 and Z_2 of the individual circuits. Two electrical circuits may be connected ***in series*** or ***in parallel***.

In a series circuit, $Z_T = Z_1 + Z_2$. In a parallel circuit, $Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}$.

Thus, to find the total impedance in a parallel circuit, you need to multiply and divide complex numbers.

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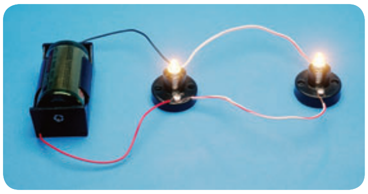
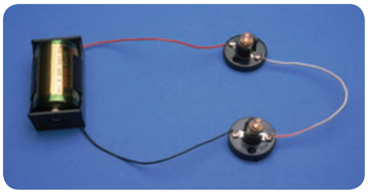
Example 4

Find the total impedance in a parallel circuit if $Z_1 = 3 + 2i$ ohms and $Z_2 = 5 - 4i$ ohms.



parallel circuit



series circuit

Solution Substitute the values of Z_1 and Z_2 into the impedance formula for parallel circuits.

$$Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$Z_T = \frac{(3+2i)(5-4i)}{? + ?}$$ Substitution

$$= \frac{? + ?i}{? - ?i}$$

$$= \frac{? + ?i}{? - ?i} \cdot \frac{? + ?i}{? + ?i}$$ Multiply numerator and denominator by the conjugate of the denominator.

$$Z_T = \frac{?}{68}$$ Definition of i and arithmetic

The total impedance is $\frac{?}{68}$ ohms.

The basic properties of inequality that hold for real numbers do not hold for nonreal complex numbers. For instance, if you were to assume $i > 0$, then multiplying both sides of the inequality by i, you would get $i \cdot i > 0 \cdot i$, or $-1 > 0$, which is not true. If you assume $i < 0$, then multiply both sides by i, you get (changing the direction) $i \cdot i > 0 \cdot i$, or again $-1 > 0$. Except for those complex numbers that are also real numbers, there are no positive or negative complex numbers.

Questions

COVERING THE IDEAS

1. **Fill in the Blank** A complex number is a number of the form $a + bi$ where a and b are ___?___ numbers.

In 2–4, give the real and imaginary parts of each complex number.

2. $5 + 17i$
3. $-4 + i\sqrt{5}$
4. i

In 5–8, rewrite the expression as a single complex number in *a* + *bi* form.

5. $(7 + 3i) - (4 - 2i)$
6. $(5 + 2i)(6 - i)$
7. $(5 + 2i)(5 - 2i)$
8. $(9 + i\sqrt{2}) + (12 - 3i\sqrt{2})$

9. What is the complex conjugate of $a + bi$?

10. Provide reasons for each step.

$(9 + 5i)(7 + 2i) = 63 + 18i + 35i + 10i^2$ a. ___?___

$= 63 + 53i + 10i^2$ b. ___?___

$= 63 + 53i + 10(-1)$ c. ___?___

$= 53 + 53i$ d. ___?___

11. Find the complex conjugate of each number.
   a. $5 + 2i$ b. $3i$ c. $-2 - 3i$ d. 4

In 12 and 13, write in *a* + *bi* form.

12. $\frac{5+2i}{4-i}$
13. $\frac{13}{2+3i}$

14. Two electrical circuits have impedances $Z_1 = 8 + 4i$ ohms and $Z_2 = 7 - 4i$ ohms. Find the total impedance if these two circuits are connected
   a. in series. b. in parallel.

15. **True or False** Every real number is also a complex number.

16. Name two fields in which complex numbers are applied.

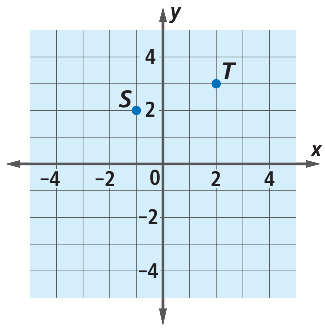
APPLYING THE MATHEMATICS

17. Write $\sqrt{-25}$ in $a + bi$ form.

18. If $Z_1 = -4 + i$ and $Z_2 = 1 - 2i$, write each expression in $a + bi$ form.
   a. $2Z_1 - Z_2$ b. $3Z_1Z_2$ c. $\frac{Z_1}{Z_2}$

19. Find two nonreal complex numbers that are not complex conjugates and whose
   a. sum is a real number. b. product is a real number.

20. A complex number $a + bi$ is graphed as the point (a, b) with the x-axis as the real axis and the y-axis as the imaginary axis.
    a. Refer to the graph at the right. S and T are the graphs of which complex numbers?
    b. Graph $S + T$.
    c. Connect (0, 0), S, $S + T$, and T to form a quadrilateral. What type of quadrilateral is formed?



21. a. Solve the equation $x^2 - 6x + 13 = 0$ using the Quadratic Formula. Write the solutions in $a + bi$ form.
    b. How are the solutions to the equation $x^2 - 6x + 13 = 0$ related to each other?

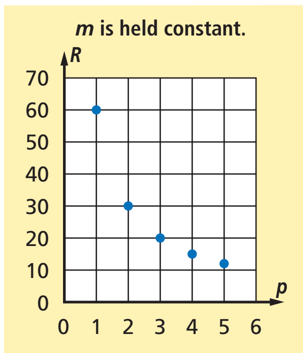
In 22 and 23, consider the complex numbers *a* + *bi* and *a* − *bi*.

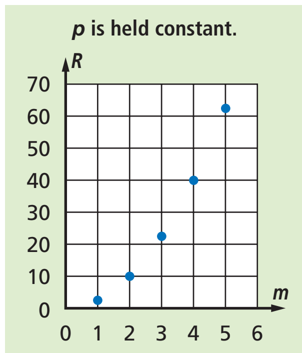
22. Find their sum and explain why it is a real number.
23. Find their product and explain why it is a real number.
24. Prove that, if two circuits connected in parallel have impedances Z_1 and Z_2 and the total impedance is Z_T, then $\frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2}$.

REVIEW

In 25 and 26, solve. Write the solutions as real numbers or multiples of *i*. (Lesson 6-8)

25. $a^2 - 3 = -8$
26. $-r^2 = 196$
27. Explain where the ± comes from in Step 8 of the derivation of the Quadratic Formula in Lesson 6-7. **(Lesson 6-2)**
28. Write a piecewise definition for the function g with equation $g(x) = |x + 2|$. **(Lessons 6-2, 3-4)**
29. The two graphs at the right show the relationships between a dependent variable R and independent variables m and p. Find an equation for R in terms of m and p. (You may leave the constant of variation as k.) **(Lesson 2-8)**





EXPLORATION

30. Refer to Question 20.
    a. Graph $z = 1 + i$ as the point (1, 1).
    b. Compute and graph z^2, z^3, and z^4.
    c. What pattern emerges? Predict where z^5, z^6, z^7, and z^8 will be.

QY ANSWERS

1. $a = -4, b = \sqrt{3}$

2. 25