Lesson

6-8 Pure Imaginary Numbers

Vocabulary

imaginary number

$\sqrt{k}$

$\sqrt{-1}$, i

pure imaginary number

BIG IDEA The square roots of negative numbers are *pure imaginary numbers* and are all multiples of $\sqrt{-1}$, defined as the number i.

Square Roots of Negative Numbers

Consider the quadratic equation $x^2 = 900$. You can solve it for x as follows.

$x^2 = 900$

$\sqrt{x^2} = \sqrt{900}$ Take square roots.

$|x| = 30$ Use the Absolute Value–Square Root Theorem.

$x = \pm 30$ Solve the absolute value equation.

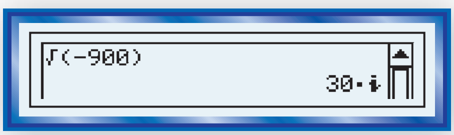
Now consider the quadratic equation $x^2 = -900$. You know that the solution cannot be a real number because the square of a real number is never negative. However, if you followed the solution above you might write:

$x^2 = -900$

$\sqrt{x^2} = \sqrt{-900}$

$|x| = ?$

So far, $\sqrt{x}$ has only been defined for $x \geq 0$. If you try to evaluate $\sqrt{-900}$ on a calculator you may see an error message, or you may see $30 \cdot i$, as shown below. The difference is whether or not your calculator is in complex number mode.



Mental Math

Solve for *x*.

a. $|x| = 4$

b. $|x - 3| = 4$

c. $|2x - 3| = 4$

A Brief History of *i*

Why does a calculator display 30 • *i* when it is in complex number mode? The symbol *i* was first used in the 18th century, but work with square roots of negative numbers started sooner. Until the 1500s, mathematicians were puzzled by square roots of negative numbers. They knew that if they solved certain quadratics, they would get negative numbers under the radical sign. However, they did not know what to do with them!

One of the first to work with these numbers was the Italian mathematician Girolamo Cardano. In a book called *Ars Magna* ("Great Art") published in 1545, Cardano reasoned as follows: When k is positive, the equation $x^2 = k$ has two solutions, $\sqrt{k}$ and $-\sqrt{k}$. If we solve the equation $x^2 = -k$ in the same way, then the two solutions are $\sqrt{-k}$ and $-\sqrt{-k}$. In this way, he defined symbols for the square roots of negative numbers. Cardano called these square roots of negatives "fictitious numbers."

Girolamo Cardano

Working in the 1600s, the French mathematician and philosopher René Descartes called them **imaginary numbers** in contrast to the numbers everyone understood, which he called "real numbers."

In his book *De Formulis Differentialibus Angularibus,* written in 1777, the Swiss mathematician Leonhard Euler wrote, "in the following I shall denote the expression $\sqrt{-1}$ by the letter i so that $i \cdot i = -1$."

Today, people around the world build on the work of Cardano, Descartes, and Euler and use the following definitions.

Definition of $\sqrt{k}$ when K Is Negative

When $k < 0$, the two solutions to $x^2 = k$ are denoted $\sqrt{k}$ and $-\sqrt{k}$.

By the definition, when $k < 0$, $\left(\sqrt{k}\right)^2 = k$. This means that we can say, for *all* real numbers r,

$$\sqrt{r} \cdot \sqrt{r} = r.$$

Suppose $k = -1$. Then we define the number i to be one of the two square roots of -1. That is, i is defined as follows.

Definition of *i*

$i = \sqrt{-1}$

Thus, i is a solution to $x^2 = -1$. The other solution is $-\sqrt{-1}$, which we call $-i$. That is, $i^2 = -1$ and $(-i)^2 = -1$.

Multiples of i, such as $7i$, are called **pure imaginary numbers**. By the definition of i, $7i = 7\sqrt{-1}$. If we assume that multiplication of imaginary numbers is commutative and associative, then

$$\begin{aligned}(7i)^2 &= 7i \cdot 7i \\ &= 7^2 \cdot i^2 \\ &= 49 \cdot (-1) \\ &= -49.\end{aligned}$$

So $7i$ is a square root of -49. We write $7i = \sqrt{-49}$ and $-7i = \sqrt{-49}$. The following theorem generalizes this result.

Square Root of a Negative Number Theorem

If $k < 0$, $\sqrt{k} = i\sqrt{-k}$.

Thus all square roots of negative numbers are multiples of i.

STOP **QY1**

QY1

Write $\sqrt{-36}$ as a multiple of i.

Example 1

Solve $x^2 = -900$.

Solution Apply the definition of $\sqrt{k}$ when k is negative.

$x = \sqrt{-900}$ or $x = -\sqrt{-900}$

Now use the Square Root of a Negative Number Theorem.

$x = i\sqrt{900}$ or $x = -i\sqrt{900}$

Simplify.

$x = 30i$ or $x = -30i$

Check Use a CAS in complex mode to solve the equation. On some machines, you use the `csolve` command to display complex solutions to equations. This CAS uses the `solve` command and gives the solution at the right. It checks.

```
solve(x^2=-900,x)
            {x=-30·i,x=30·i}
```

Example 2

a. Show that $i\sqrt{5}$ is a square root of -5.

b. What is the other square root of -5?

Solution

a. Multiply $i\sqrt{5}$ by itself. Assume the Commutative and Associative Properties.

$$i\sqrt{5} \cdot i\sqrt{5} = i \cdot i \cdot \sqrt{5} \cdot \sqrt{5}$$
$$= i^2 \cdot 5$$
$$= -1 \cdot 5$$
$$= -5$$

b. Take the opposite of the square root from Part a. The other square root of –5 is $-i\sqrt{5}$.

STOP QY2

QY2

Use a CAS to show that $-i\sqrt{5}$ is a square root of –5.

Due to the long history of quadratic equations, solutions to them are described in different ways. The following all refer to the same numbers.

the solutions to $x^2 = -5$

the square roots of –5

$\sqrt{-5}$ and $-\sqrt{-5}$

$i\sqrt{5}$ and $-i\sqrt{5}$

The last two forms could also be written $\sqrt{5}i$ and $-\sqrt{5}i$ as a CAS displays them. On handwritten materials and in textbooks you commonly see $i\sqrt{5}$ to clearly show that the i is not underneath the radical sign.

Operations with Pure Imaginary Numbers

The Commutative, Associative, and Distributive Properties of Addition and Multiplication are true for all imaginary numbers, as are all theorems based on these postulates. Consequently, you can use them when working with multiples of i, just as you would when working with multiples of any real numbers.

Example 3

Simplify the following.

a. $(5i)(3i)$ b. $\sqrt{-16} - \sqrt{-64}$ c. $\sqrt{-2} + \sqrt{-2}$ d. $\dfrac{\sqrt{-100}}{\sqrt{-81}}$

Solution

a. $(5i)(3i) = 15i^2 = 15 \cdot -1 = -15$

b. $\sqrt{-16} - \sqrt{-64} = 4i - 8i = -4i$

c. $\sqrt{-2} + \sqrt{-2} = i\sqrt{2} + i\sqrt{2} = 2i\sqrt{2}$

d. $\dfrac{\sqrt{-100}}{\sqrt{-81}} = \dfrac{10i}{9i} = \dfrac{10}{9}$

In operating with imaginary numbers in radical form, be sure to follow the order of operations, treating the square root as a grouping symbol.

Activity

MATERIALS calculator

Work with a partner to calculate with imaginary numbers.

Step 1 Make sure that your calculator is in complex mode. Use your calculator to write each answer in $a + bi$ form.

$(4i)(25i)$	$4i - 5i$	$\sqrt{-4} - \sqrt{-25}$	$\frac{\sqrt{-4}}{\sqrt{25}}$	$\sqrt{-4} \cdot \sqrt{25}$	$\sqrt{-4} \cdot \sqrt{-25}$
?	?	?	?	?	?

Step 2 Look for patterns in your results from Step 1. When are the results real numbers and when are they imaginary?

Step 3 Calculate $\sqrt{4} \cdot \sqrt{25}$ and $\sqrt{4 \cdot 25}$. Do your results support the property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for nonnegative real numbers? Why or why not?

Step 4 Calculate $\sqrt{-4 \cdot 25}$ and $\sqrt{-4 \cdot -25}$. Now look at your Step 1 results for $\sqrt{-4} \cdot \sqrt{25}$ and $\sqrt{-4} \cdot \sqrt{-25}$. Does $\sqrt{a}\sqrt{b} = \sqrt{ab}$ if either a or b, but not both, is a negative real number? Does $\sqrt{a}\sqrt{b} = \sqrt{ab}$ if both a and b are both negative real numbers? Why or why not?

In the Activity, $\sqrt{-4 \cdot -25} = \sqrt{100} = 10$. You can verify that $\sqrt{-4} \cdot \sqrt{-25} = 10$ by hand as follows.

$$\begin{aligned}\sqrt{-4} \cdot \sqrt{-25} &= i\sqrt{4} \cdot i\sqrt{25}\\ &= 2 \cdot 5 \cdot i \cdot i\\ &= 10i^2\\ &= -10\end{aligned}$$

So $\sqrt{-4 \cdot -25} \neq \sqrt{-4} \cdot \sqrt{-25}$ is a counterexample showing that the property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ *does not hold* for *nonnegative* real numbers a and b.

Questions

COVERING THE IDEAS

1. **True or False** All real numbers have square roots.
2. Write the solutions to $x^2 = -4$.

3. **Multiple Choice** About when did mathematicians begin to use roots of negative numbers as solutions to equations?
   A sixth century B twelfth century
   C sixteenth century D twenty-first century
4. Who first used the term *imaginary number*?
5. Who was the first person to suggest using i for $\sqrt{-1}$?

In 6 and 7, True or False.

6. For all real numbers x, $\sqrt{-x} < 0$.
7. If $b > 0$, $\sqrt{-b} = i\sqrt{b}$.

In 8 and 9, solve for x.
   a. Write the solutions to each equation with a radical sign.
   b. Write the solutions without a radical sign.

8. $x^2 + 25 = 0$
9. $x^2 + 16 = 0$
10. **True or False** $i\sqrt{7}$ is a square root of -7. Justify your answer.
11. Show that $-6i$ is a square root of -36.
12. Show that $\sqrt{-3}\sqrt{-27} \neq \sqrt{81}$.

In 13–15, simplify.

13. $\sqrt{-53}$
14. $\sqrt{-121}$
15. $-\sqrt{72}$

In 16–21, perform the indicated operations. Give answers as real numbers or multiples of i.

16. $-2i + 7i$
17. $10i - 3i$
18. $(4i)(17i)$
19. $3\sqrt{-16} + \sqrt{-64}$
20. $\frac{\sqrt{-81}}{\sqrt{-9}}$
21. $-\sqrt{-49} + \sqrt{-49}$

In 22–24, simplify the product.

22. $\sqrt{-5} \cdot \sqrt{-5}$
23. $\sqrt{-10} \cdot \sqrt{-30}$
24. $\sqrt{3} \cdot \sqrt{-3}$
25. a. Does $\sqrt{-3 \cdot 27} = \sqrt{-3} \cdot \sqrt{27}$?
    b. Does your answer in Part a provide a counterexample to $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for $a < 0$ and $b > 0$?
26. For what real number values of x and y does $\sqrt{xy} \neq \sqrt{x}\sqrt{y}$?

APPLYING THE MATHEMATICS

In 27 and 28, simplify.

27. $\sqrt{-477{,}481}$
28. $\frac{5i + 9i}{2i}$

In 29 and 30, True or False. If false, give a counterexample.

29. The sum of any two imaginary numbers is imaginary.

30. The product of any two imaginary numbers is imaginary.

31. Verify your solutions to $x^2 = -4$ in Question 2 by using the Quadratic Formula to solve $x^2 + 0x + 4 = 0$.

32. Solve $9y^2 + 49 = 0$ using the Quadratic Formula.

In 33 and 34, solve the equation.

33. $a^2 + 12 = 5$

34. $(b - 5)^2 + 13 = 9$

REVIEW

In 35 and 36, solve for x using the Quadratic Formula. (Lesson 6-7)

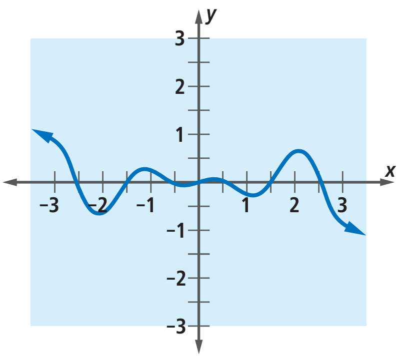
35. $8x^2 - 2x - 15 = 0$

36. $x^2 + dx - 2d^2 = 0$

37. Use quadratic regression to find an equation of the parabola that contains the points (2, 523.3), (4, 1126.3), and (8, 2338.3). (Lesson 6-6)

38. A chef reports that with 5 kilograms of flour, 12 loaves of bread and 6 pizza crusts can be made. With 2 kilograms of flour, 1 loaf of bread and 10 pizza crusts can be made. How much flour is needed to make one loaf of bread? How much is needed for one pizza crust? (Lesson 5-4)

39. Consider the function graphed below. Give its domain and range. (Lesson 1-4)



EXPLORATION

40. By definition, $i^2 = -1$. So $i^3 = i^2 \cdot i = -1 \cdot i = -i$ and $i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1$. Continue this pattern to evaluate and simplify each of i^5, i^6, i^7, and i^8. Generalize your result to predict the value of i^{2009}, i^{2010}, and i^{2020}. Explain how to simplify any positive power of i.

QY ANSWERS

1. $6i$

2.