Lesson 6-7 The Quadratic Formula

Vocabulary

Quadratic Formula

BIG IDEA The *Quadratic Formula* gives the solutions to any quadratic equation in standard form whose coefficients are known.

Mental Math

Expand the binomial.

a. $(a - b)^2$

b. $(a - 3b)^2$

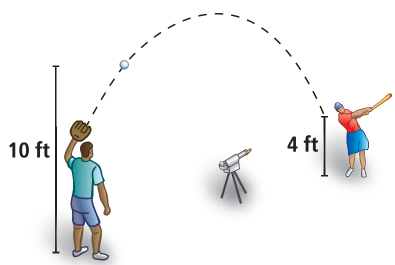
c. $(5a - 3b)^2$

d. $(5a^4 - 3b^6)^2$

To help train outfielders, the coach of a baseball team (who is also a math teacher) uses a hitting machine. The machine hits a 4-foot-high pitch, and the ball travels toward the outfield along a nearly parabolic path. Let x be the distance along the ground (in feet) of the ball from home plate, and h be the height (in feet) of the ball at that distance. Using estimated heights of the ball at a various points along its path, he found the following regression equation to model the flight of the ball.

$$h = -0.00132x^2 + 0.545x + 4$$

If an outfielder leaps and catches a ball 10 feet off the ground, how far is he from home plate? To answer this, you can solve the flight equation when $h = 10$.



$$10 = -0.00132x^2 + 0.545x + 4$$

Subtract 10 from each side to rewrite the equation in standard form.

$$0 = -0.00132x^2 + 0.545x - 6$$

You could solve this equation by rewriting it in vertex form, but in previous courses you have probably used a formula that gives the solutions. The **Quadratic Formula** is a theorem that can be proved by completing the square, starting with a general quadratic equation in standard form.

Quadratic Formula Theorem

If $ax^2 + bx + c = 0$ and $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The proof is given on the next page.

Given the equation $ax^2 + bx + c = 0$, where $a \neq 0$.

Proof

1.	$x^2 + \frac{b}{a}x + \frac{c}{a} = \frac{0}{a}$	Divide both sides by a so the coefficient of x^2 is 1.
2.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$	Add $-\frac{c}{a}$ to each side.
3.	$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$	Complete the square by adding $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ to both sides.
4.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$	Write the left side as a binomial squared.
5.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Add the fractions on the right side.
6.	$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}}$	Take the square roots of both sides.
7.	$\left\lvert x + \frac{b}{2a} \right\rvert = \sqrt{\frac{b^2 - 4ac}{4a^2}}$	Use the Absolute Value–Square Root Theorem.
8.	$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Use the definition of absolute value.
9.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Add $-\frac{b}{2a}$ to both sides.

If you solve $ax^2 + bx + c = 0$ on a CAS, it is likely to display the solutions in a compound sentence.

$$\text{solve}(a \cdot x^2 + b \cdot x + c = 0, x)$$
$$x = \frac{\sqrt{b^2 - 4 \cdot a \cdot c} - b}{2 \cdot a} \text{ or } x = \frac{-\left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)}{2 \cdot a}$$

Using the Quadratic Formula

Example 1

To the nearest foot, how far from home plate was the outfielder when he leaped to catch the 10-ft high ball?

Solution Use the Quadratic Formula on the equation $0 = -0.00132x^2 + 0.545x - 6$.

Here $a = -0.00132$, $b = 0.545$, and $c = -6$.

$$x = \frac{-0.545 \pm \sqrt{(0.545)^2 - 4(-0.00132)(-6)}}{2(-0.00132)}$$

(continued on next page)

A calculator can approximate the two solutions in one step. Here are the intermediate steps

$$\approx \frac{-0.545 \pm \sqrt{0.265}}{-0.00264}$$

Estimate the square root and separate the solutions.

$$x \approx \frac{-0.545 + 0.515}{-0.00264} \quad \text{or} \quad x \approx \frac{-0.545 - 0.515}{-0.00264}$$

$$x \approx 11 \text{ ft} \quad \text{or} \quad x \approx 402 \text{ ft}$$

The ball reaches a height of 10 feet in two places. The first is when the ball is on the way up and about 11 feet away from home plate. The second is when the ball is on the way down and about 402 feet from home plate. Between these distances the ball is over 10 feet high. An outfielder is unlikely to be 11 feet from home plate, so he was about 402 feet away.

Check Use the `solve` command on a CAS.

$\text{solve}(-.00132 \cdot x^2 + .545 \cdot x - 6 = 0, x)$

$x = 11.3195 \text{ or } x = 401.559$

In Example 1, the number $b^2 - 4ac$ is not a perfect square. When this is the case, there are no rational number solutions to the equation. When $b^2 - 4ac$ is a perfect square, as it is in Example 2, the solutions are always rational.

GUIDED

Example 2

Solve $5x^2 + 13x - 6 = 0$.

Solution Use the Quadratic Formula.

$a = \underline{\ ?\ }$ $\quad b = \underline{\ ?\ }$ $\quad c = \underline{\ ?\ }$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(\ ?\ ) \pm \sqrt{(\ ?\ )^2 - 4(\ ?\ )(\ ?\ )}}{2(\ ?\ )}$$

$$x = \frac{? \pm \sqrt{?}}{?}$$

So, $x = \frac{? + ?}{?}$ $\quad$ or $\quad$ $x = \frac{? - ?}{?}$

$x = \underline{\ ?\ }$ $\quad$ or $\quad$ $x = \underline{\ ?\ }$

A quadratic equation must be in standard form before the Quadratic Formula can be applied.

Example 3

Recall that the explicit formula for the sequence t_n of triangular numbers is $t_n = \frac{n(n+1)}{2}$. Is 101,475 a triangular number? If it is, which term of the sequence is it?

Solution Set $t_n = 101{,}475$ and solve for n.

$$t_n = \frac{n(n+1)}{2} = 101{,}475$$

Put the equation in standard form.

$n(n+1) = 202{,}950$	Multiply both sides by 2.
$n^2 + n = 202{,}950$	Expand.
$n^2 + n - 202{,}950 = 0$	Add –202,950 to both sides.

Use the Quadratic Formula with $a = 1$, $b = 1$, and $c = -202{,}950$.

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-202{,}950)}}{2(1)}$$

$$n = \frac{-1 \pm \sqrt{811{,}801}}{2}$$

$$n = \frac{-1 \pm 901}{2}$$

$$n = -451 \text{ or } n = 450$$

Because 450 is a positive integer, 101,475 is the 450th triangular number.

STOP QY

> **QY**
>
> Use the explicit formula for the triangular numbers to check the solution to Example 3.

Questions

COVERING THE IDEAS

1. If $ax^2 + bx + c = 0$, and $a \neq 0$, write the two values of x in terms of a, b, and c as a compound sentence.

In 2–4 refer to the proof of the Quadratic Formula.

2. Why must a be nonzero in the Quadratic Formula?
3. Why is it necessary to divide both sides by a in the first step?
4. Write $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$ as the square of a binomial.

In 5 and 6, a quadratic equation is given. Solve each using the Quadratic Formula.

5. $10z^2 + 13z + 3 = 0$
6. $2n^2 - 11n + 12 = 0$

In 7–10, consider the equation $h = -0.00132x^2 + 0.545x + 4$ from the beginning of the lesson. The left field wall at Fenway Park in Boston is 37 ft tall and 310 ft from home plate.

7. What do x and h represent?
8. Would a ball on the path of this example be a home run? (That is, would it have gone over the left field wall?)
9. Trace a graph of the equation to find the maximum height reached by the ball.
10. How far from home plate would the ball hit the ground, if the outfielder missed it?
11. Refer to Example 2. How do your solutions change when you solve $5x^2 - 13x - 6 = 0$ instead?

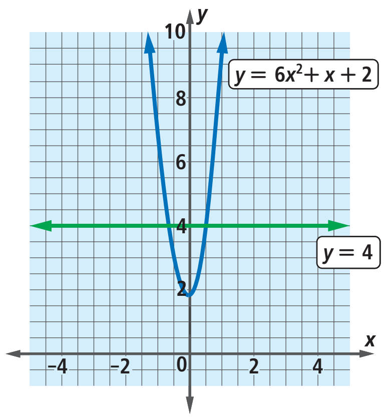
Fenway Park, home to the Boston Red Sox

In 12 and 13, refer to Example 3.

12. Use a CAS to check the solution for Example 3.
13. Show that 10,608 is not a triangular number.

APPLYING THE MATHEMATICS

14. What is another way to solve $ax^2 + bx + c = 0$, $a \neq 0$, besides using the Quadratic Formula or a CAS?
15. Consider this sequence of quadratic equations.
    Q_1: $x^2 + 5x + 6 = 0$
    Q_2: $x^2 + 7x + 12 = 0$
    Q_3: $x^2 + 9x + 20 = 0$
    Q_4: $x^2 + 11x + 30 = 0$
    Q_5: $x^2 + 13x + 42 = 0$
    a. Solve each quadratic equation.
    b. Find the product and the sum of each set of answers in Part a.
    c. What is the connection between the solutions to the quadratic equations and the coefficients of the quadratic equations?
16. Consider the parabola with equation $y = m^2 - 2m - 7$.
    a. Find the values of m for which $y = 0$. What are these values called?
    b. Find the vertex of this parabola.
    c. Give an equation for its axis of symmetry.
    d. Sketch a graph of the parabola.
17. The graphs of $y = 6x^2 + x + 2$ and $y = 4$ are shown at the right. Use the Quadratic Formula to find the points of intersection.



18. **Fill in the Blanks** Given $ax^2 + bx + c = 0$, where $a \neq 0$, provide the missing algebra in this alternate proof of the Quadratic Formula that does not use completing the square.

a. Multiply both sides by $4a$. $\underline{\quad ? \quad}$

Recognize that the terms $4a^2x^2 + 4abx$ are initial terms of the binomial square $(2ax + b)^2$.

b. Replace $4a^2x^2 + 4abx$ with $(2ax + b)^2 - b^2$. (Explain why $-b^2$ is added in this step.) $\underline{\quad ? \quad}$

c. Add $b^2 - 4ac$ to both sides. $(2ax + b)^2 = b^2 - 4ac$

d. Take the square roots of both sides. $\underline{\quad ? \quad}$

e. Use the Absolute Value-Square Root Theorem. $|2ax + b| = \sqrt{b^2 - 4ac}$

f. Use the definition of absolute value. $\underline{\quad ? \quad}$

g. Add $-b$ to both sides. $2ax = -b \pm \sqrt{b^2 - 4ac}$

h. Divide both sides by $2a$. $\underline{\quad ? \quad}$

REVIEW

19. Solve a system of equations to write an equation for the parabola that contains the points $(-1, -7)$, $(0, 3)$, and $(3, 9)$. **(Lesson 6-6)**

20. A ball is thrown straight up from a height of 1.5 meters with initial upward velocity of $6\frac{\text{m}}{\text{s}}$. Find the maximum height of the ball and the time the ball takes to reach this height. **(Lessons 6-5, 6-4)**

21. Let $M = \begin{bmatrix} 4 & 12 \\ -8 & x \end{bmatrix}$. For what value or values of x does M^{-1} not exist? **(Lesson 5-5)**

22. State the domain and range of the function f when $f(x) = -\frac{\sqrt{2}}{x}$. **(Lesson 2-6)**

23. a. Let $g_n = 1 + (-1)^n$. Calculate g_1, g_2, g_3, and g_4.

b. Describe how to quickly know g_n for any positive integer n.

c. What is g_{7532}? **(Lesson 1-8)**

EXPLORATION

24. Find out what you can about the formula used to solve the general cubic equation $ax^3 + bx^2 + cx + d = 0$.

QY ANSWER

Does $t_{450} = \frac{450(450 + 1)}{2} =$ 101,475? $\frac{450 \cdot 451}{2} = \frac{202{,}950}{2}$ $= 101{,}475$. It checks.