6-5

Completing the Square

▶ **BIG IDEA** By adding a number to an expression of the form $x^2 + bx$, you can create a new expression that is the square of a binomial.

You have now seen two forms for an equation of a parabola.

Standard form $\qquad y = ax^2 + bx + c$

Vertex form $\qquad y - k = a(x - h)^2$ or $y = a(x - h)^2 + k$

Because each form is useful, being able to convert between forms is helpful. In Lesson 6–4 you saw how to convert vertex form to standard form. In this lesson, you will see how to convert standard form to vertex form.

What Is Completing the Square?

One method for converting standard form to vertex form is called **completing the square**. Remember that $(x + h)^2 = x^2 + 2hx + h^2$. The trinomial $x^2 + 2hx + h^2$ is called a **perfect-square trinomial** because it is the square of a binomial. At the right you can see that $x^2 + 2hx + h^2$ is the area of a square with side length $x + h$.

	x	h
h	hx	h^2
x	x^2	hx
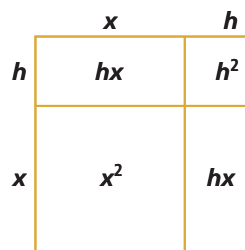
Area $= (x + h)^2 = x^2 + 2hx + h^2$

Mental Math

How many solutions does a 2 × 2 system of linear equations have if the two equations in the system represent

a. oblique lines with different slopes?

b. one vertical line and one nonvertical line?

c. the same line?

**d.** parallel lines with different y–intercepts?

Example 1

a. What number should be added to $x^2 + 6x$ to make a perfect–square trinomial?

b. Write the perfect–square trinomial as the square of a binomial.

(continued on next page)

Solution 1 Use geometry.

a. Draw a picture to represent $x^2 + 6x + \underline{\,?\,}$. Since the sum of the areas of the two rectangles that are not squares must be $6x$, the area of each rectangle is $3x$. Think: What is the area of the missing square in the upper right corner that allows you to complete the larger square? (This is the reason this process is called "completing the square".) A square with area 9 would complete the larger square.

So, 9 must be added to $x^2 + 6x$ to make a perfect-square trinomial.

b. In the picture at the right, the length of the side of the square is $x + 3$. So, $x^2 + 6x + 9 = (x + 3)^2$.

Area $= x^2 + 6x + \underline{\quad ? \quad}$
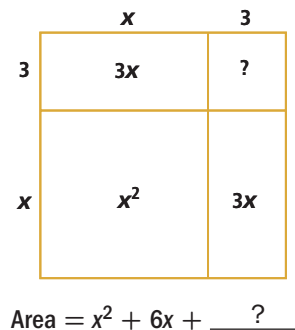
Solution 2 Use algebra.

a. Compare $x^2 + 6x + \underline{\,?\,}$ with the perfect-square trinomial $x^2 + 2hx + h^2$. The first terms, x^2, are identical. To make the second terms equal, set

$$6x = 2hx.$$
So, $\qquad h = 3.$

The term added to make a perfect-square trinomial should be h^2, or 9.

b. Since $x^2 + 2xh + h^2 = (x + h)^2$, and you found in Part a that $h = 3$,
$x^2 + 6x + 9 = (x + 3)^2$.

Check Apply the Binomial Square Theorem to expand $(x + 3)^2$.

$(x + 3)^2 = x^2 + 6x + 9$. It checks.

To generalize Example 1, consider the expression $x^2 + bx + \underline{\,?\,}$. What goes in the blank so that the result is a perfect-square trinomial?

$$x^2 + bx + \underline{\,?\,} = x^2 + 2hx + h^2$$

Because $b = 2h$, $h = \frac{b}{2}$. Then $h^2 = \left(\frac{b}{2}\right)^2$. This illustrates the following theorem.

Completing the Square Theorem

To complete the square on $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.

Proof $x^2 + bx + \left(\frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2$

🛑 **QY**

▶ **QY**

What number should be added to $x^2 - 24x$ to make a perfect square trinomial?

Completing the Square to Find the Vertex of a Parabola

The Completing the Square Theorem can be used to transform an equation of a parabola from standard form into vertex form.

GUIDED

Example 2

a. Rewrite the equation $y = x^2 + 12x + 3$ in vertex form.

b. Find the vertex of the parabola.

Solution

a. Rewrite the equation so that only terms with x are on one side.

$$y - 3 = x^2 + 12x$$

Use the Completing the Square Theorem. Here,

$$b = \underline{\quad?\quad}, \text{ so } \left(\frac{b}{2}\right)^2 = \underline{\quad?\quad}.$$

Complete the square on $x^2 + 12x$.

$y - 3 + \underline{\;?\;} = x^2 + 12x + \underline{\;?\;}$ Add $\left(\frac{b}{2}\right)^2$ to both sides.

$y + \underline{\quad?\quad} = x^2 + 12x + \underline{\;?\;}$ Simplify the left side.

$y + \underline{\quad?\quad} = (x + \underline{\;?\;})^2$ Apply the Binomial Square Theorem.

b. We can read the vertex from the vertex form of the equation. The vertex of the parabola is ($\underline{\;?\;}$, $\underline{\;?\;}$).

Check

a. Graph $y = x^2 + 12x + 3$ and the vertex form from Part a on the same set of axes in an appropriate window. Trace and toggle between the graphs for several values of x. They should be identical.

b. Trace to estimate the vertex on the graph.
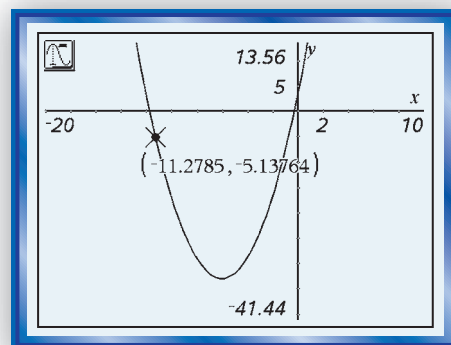
Equating Expressions to Find the Vertex of a Parabola

Example 2 involves a parabola in which the coefficient of x^2 is 1. Example 3 shows how to find the vertex of a parabola if the coefficient of x^2 is not 1. This kind of expression occurs in describing heights of thrown objects and paths of projectiles.

Example 3

Suppose a ball is thrown straight up from a height of 12 feet with an initial velocity of 32 $\frac{ft}{sec}$. Then the ball's height y after x seconds is given by the formula $y = -16x^2 + 32x + 12$.

a. Rewrite the formula in the vertex form of the equation for a parabola.

b. Find the maximum height of the ball and the time it takes for the ball to reach that point.

Solution

a. Equate the given expression for y with the general vertex form.

$$-16x^2 + 32x + 12 = a(x - h)^2 + k$$

Enter this equation into a CAS. The CAS will automatically expand the square of the binomial, as shown here.

$-16 \cdot x^2 + 32 \cdot x + 12 = a \cdot (x - h)^2 + k$

$-16 \cdot x^2 + 32 \cdot x + 12 = a \cdot x^2 - 2 \cdot a \cdot h \cdot x + a \cdot h^2 + k$

The coefficients of x^2 on the two sides must be equal, so $a = -16$. Substitute this value for a.

$+ 32 \cdot x + 12 = a \cdot x^2 - 2 \cdot a \cdot h \cdot x + a \cdot h^2 + k | a = -16$

$-16 \cdot x^2 + 32 \cdot x + 12 = -16 \cdot x^2 + 32 \cdot h \cdot x - 16 \cdot h^{2}$

Add $16x^2$ to both sides.

$\cdot x + 12 = -16 \cdot x^2 + 32 \cdot h \cdot x - 16 \cdot h^2 + k) + 16 \cdot x^2$

$32 \cdot x + 12 = 32 \cdot h \cdot x - 16 \cdot h^2 + k$

Equate the coefficients of x to see that $32 = 32h$, so $h = 1$. Substitute this value for h.

$32 \cdot x + 12 = 32 \cdot h \cdot x - 16 \cdot h^2 + k | h = 1$

$32 \cdot x + 12 = 32 \cdot x + k - 16$

Add $-32x$ to both sides. Then add 16 to both sides. The display at the right show that $k = 28$.

Substitute back for a, h, and k in the square form.

$$y = -16(x - 1)^2 + 28, \text{ or}$$
$$y - 28 = -16(x - 1)^2$$

$(32 \cdot x + 12 = 32 \cdot x + k - 16) + -32 \cdot x \qquad 12 = k - 16$

$(12 = k - 16) + 16 \qquad 28 = k$

b. The vertex of this parabola is (1, 28).
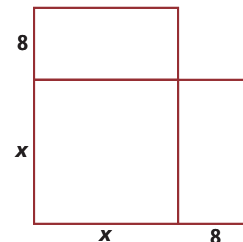So the maximum height of the ball is 28 feet, 1 second after it is thrown.

Check Graph $y = -16x^2 + 32x + 12$. Trace to estimate the maximum value of the parabola. It checks.
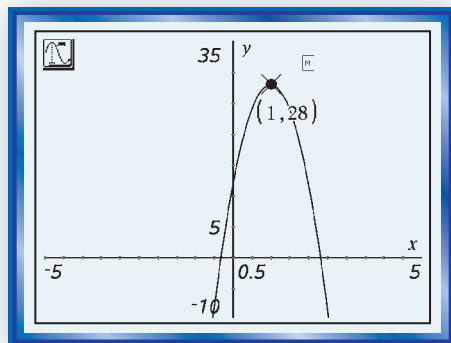
Questions

COVERING THE IDEAS

1. **a.** Give the sum of the areas of the three rectangles at right.
   b. What number must be added to this sum to complete the square?
   c. **Fill in the Blanks** Complete the equation
   $x^2 + \underline{}\,x + \underline{} = (x + \underline{})^2$.

Fill in the Blanks In 2–5, find a number to make the expression a perfect-square trinomial.

2. $x^2 + 14x + \underline{}$
3. $n^2 - n + \underline{}$
4. $z^2 - 30z + \underline{}$
5. $t^2 + \frac{2}{3}t + \underline{}$

In 6 and 7, an equation in standard form is given.
   a. Rewrite the equation in vertex form.
   b. Find the vertex of the parabola represented by each equation.

6. $y = x^2 + 4x + 11$
7. $y = -2x^2 - 6x - 14$

8. Refer to Example 3. Generate a table of values for the equation $h = -16t^2 + 32t + 12$, or fill in a table like the one below by hand.

t	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2
h	?	?	?	?	?	?	?	?	?

   a. The points in the table are symmetric to a vertical line through a particular point in the table. Which point?
   b. How does your answer to Part a compare to the vertex found in Example 3?

APPLYING THE MATHEMATICS

9. Suppose a ball is thrown straight up from a height of 8 feet with an initial upward velocity of 64 $\frac{\text{ft}}{\text{sec}}$.

   a. Write an equation to describe the height h of the ball after t seconds.

   b. How high is the ball after 1 second?

   c. Determine the maximum height attained by the ball by completing the square.

   d. Sketch a graph your equation from Part a.

   e. How long will it take for the ball to land on the ground?

10. **Fill in the Blank** What is the missing term in the expression $x^2 + \frac{b}{a}x + \underline{\ ?\ }$ if the expression is a perfect-square trinomial?

11. Find an equation in vertex form equivalent to $y = 5x^2 - 2x + 15$.

In 12 and 13, consider the following. When a quadratic function is graphed, the second coordinate of the vertex of the parabola is always the minimum or maximum value of the function. Commands on some calculators may help you find those values. On one calculator these commands are `fMin` and `fMax`.

12. In the display at the right, `fMin` and `fMax` have been calculated. You are given the x-coordinate of the minimum point of the graph of $y = 3x^2 - 12x + 14$ and the x-coordinate of the maximum point of the graph of $y = -10x^2 + 60x$. Find the coordinates of the vertex of each parabola.

$\text{fMin}\left(3 \cdot x^2 - 12 \cdot x + 14, x\right)$	$x=2$
$\text{fMax}\left(-10 \cdot x^2 + 60 \cdot x, x\right)$	$x=3$

13. At the right, why is the x-coordinate of the minimum value stated as positive or negative infinity?

$\text{fMin}\left(x^2 - 2 \cdot x + 1, x\right)$	$x=\infty$ or $x=-\infty$

14. You run Twin Wheels bike-rental shop. You currently charge $10 per day and average 56 rentals a day. In researching a price increase, you believe that for every fifty-cent increase in rental price you can expect to lose two rentals a day. Let n = the number of fifty-cent increases.

   a. Write an expression for the new price after n increases.

   b. Write an expression for the expected number of rentals after n increases.

   c. The total income for the day is equal to the price times the number of rentals. Multiply the expressions in Parts a and b to get an expression for the total daily income.

   d. Find the rental price that will maximize the total daily income.

REVIEW

15. Jailah tosses a ball upward from an initial height of 1.6 meters. The ball lands on the ground 4.8 seconds later. What was the upward velocity of Jailah's throw? (**Lesson 6–4**)

16. Rewrite the equation $y = 4(3 - x)^2 - 8$ in standard form. (**Lesson 6–4**)

17. On the coordinate grid at the right are the parabola with equation $y = -x^2$ and its image under a translation. (**Lessons 6–3, 4–10**)

    a. What translation maps the parabola with equation $y = -x^2$ onto the other parabola?

    b. Write an equation for the image.
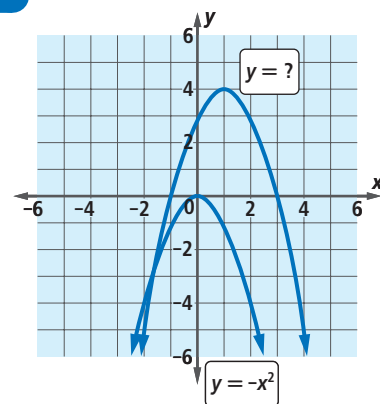
18. Solve the system $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -9 \end{bmatrix}$. (**Lesson 5–6**)

19. The table at the right shows the growth of the European Union (EU) since its inception in 1958. (**Lesson 3–5**)

    a. Draw a scatterplot of this data.

    b. Find an equation for the regression line, and draw the line on your scatterplot.

    c. How many member states does your regression line predict for the year 2058, the 100th anniversary of the founding of the EU?

Years since 1958	Number of Member States
0	6
15	9
23	10
28	12
37	15
46	25
49	27

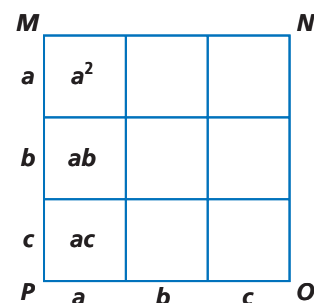European Union member states
Non-member states

In 20 and 21, find x. (**Previous Course**)

20. $3^{2x} \cdot 3^4 = 3^{24}$

21. $(2^x)^5 = 2^{25}$

EXPLORATION

22. *MNOP* is a square with sides of length $a + b + c$. The areas of three regions inside *MNOP* are shown in the diagram.

    a. Find the areas of the other six regions.

    b. Use the drawing to help you expand $(a + b + c)^2$.

    c. Make a drawing to illustrate the expansion of $(a + b + c + d)^2$.

M — N

	a		
a	a^2		
b	ab		
c	ac		

P   a   b   c   O

QY ANSWER

144