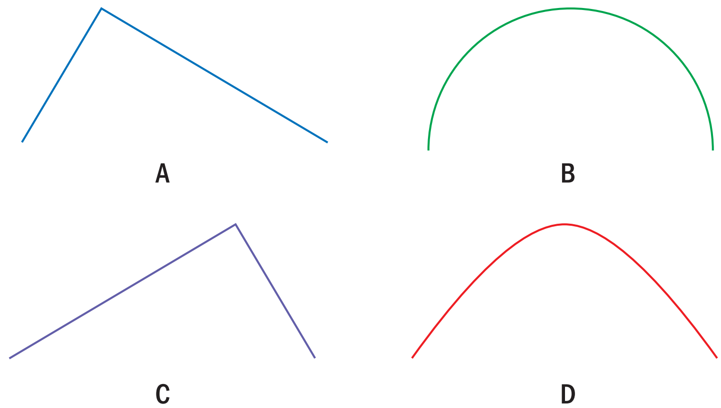
Lesson
6-4 The Graph of $y = ax^2 + bx + c$

BIG IDEA The graph of $y = ax^2 + bx + c$, $a \neq 0$, is a parabola that opens upward if $a > 0$ and downward if $a < 0$.

Standard Form for the Equation of a Parabola

Homer King hits a high-fly ball to deep center field. Ignoring air currents, which curve below most closely resembles the flight path of the ball?



The answer is D, because high-fly balls and many other projectiles travel in parabolic paths. These paths have equations that can be put into the standard form of a quadratic function, $y = ax^2 + bx + c$. In general, any equation for a parabola that can be written in the vertex form $y - k = a(x - h)^2$ can be rewritten in the standard form $y = ax^2 + bx + c$.

Mental Math

Give an example of an equation whose graph contains (1, 3) and is

a. a line.

b. a hyperbola.

c. a parabola.

d. not a line, hyperbola, or parabola.

Example 1

Show that the equation $y - 16 = 3(x - 5)^2$ can be rewritten in the form $y = ax^2 + bx + c$, and give the values of a, b, and c.

Solution Solve for y, then expand the binomial, distribute, and simplify.

(continued on next page)

$y - 16 = 3(x - 5)^2$

$y = 3(x - 5)^2 + 16$ Add 16 to both sides.

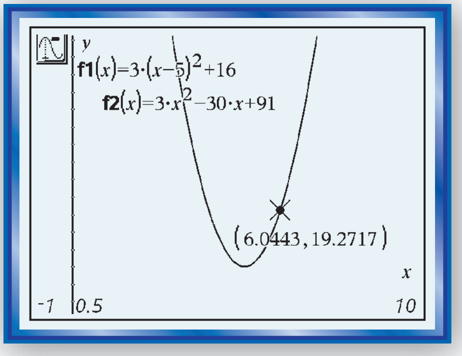
$y = 3(x^2 - 10x + 25) + 16$ Expand the binomial square.

$y = 3x^2 - 30x + 75 + 16$ Distribute the 3.

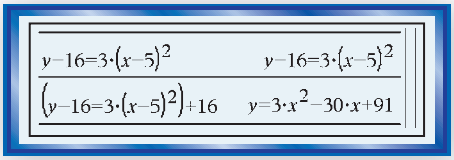
$y = 3x^2 - 30x + 91$ Arithmetic

So the original equation is equivalent to one in standard form with $a = 3$, $b = -30$, and $c = 91$.

Check 1 Graph both $y = 3(x - 5)^2 + 16$ and $y = 3x^2 - 30x + 91$ on your graphng utility. Use the trace feature and toggle between graphs to see if the coordinates match. The graphs seem to be identical.



Check 2 Enter the original equation into a CAS. Add 16 to both sides of the equation.



This CAS expands the right side automatically. It checks.

QY1

In general, to change vertex form to standard form, solve for y and expand.

$y - k = a(x - h)^2$

$y = a(x - h)^2 + k$ Add k to each side.

$y = a(x^2 - 2hx + h^2) + k$ Square the binomial.

$y = ax^2 - 2ahx + ah^2 + k$ Use the Distributive Property.

This is in standard form, with $b = -2ah$ and $c = ah^2 + k$. With these substitutions, the equation becomes

$$y = ax^2 + bx + c.$$

> **QY1**
>
> In Example 1, subtract the final expression for y from the original expression for y. What do you get?

Congruent Parabolas

Because the parabola determined by the equation $y - k = a(x - h)^2$ is a translation image of the parabola determined by the equation $y = ax^2$, the two parabolas are congruent. For all h and k, $y - k = a(x - h)^2$ can be written in standard form, so we have the following theorem.

Parabola Congruence Theorem

The graph of the equation $y = ax^2 + bx + c$ is a parabola congruent to the graph of $y = ax^2$.

Recall that a *quadratic function* is any function f whose equation can be put in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. Thus, the graph of every quadratic function is a parabola, with y-intercept $f(0) = c$. Unless otherwise specified, the domain of a quadratic function is the set of real numbers. When $a > 0$, the range is the set of real numbers greater than or equal to its minimum value. When $a < 0$, the range is the set of real numbers less than or equal to its maximum value.

Applications of Quadratic Functions

Some applications of quadratic functions have been known for centuries. In the early 17th century, Galileo described the height of an object in free fall. Later that century, Isaac Newton derived his laws of motion and the law of universal gravitation. In developing his mathematical equations for the height of an object, Newton reasoned as follows:

- Gravity is a force that pulls objects near Earth downward. Without gravity, a ball thrown upward would continue traveling at a constant rate. Then its height would be (initial height) + (upward velocity) • (time). So, if it were thrown at 59 feet per second from an initial height of 4 feet, it would continue traveling at 59 feet per second, and its height after t seconds would be $4 + 59t$.
- Galileo had shown that gravity pulls the ball downward a total of $16t^2$ feet after t seconds. This effect can be subtracted from the upward motion without gravity. Therefore, after t seconds, its height in feet would be $4 + 59t - 16t^2$ feet. The number 16 in the expression is a constant for all objects falling at or near Earth's surface when the distances are measured in feet. When measured in meters, this number is 4.9.

Sir Isaac Newton

Example 2

A thrown ball has height $h = -16t^2 + 59t + 4$ after t seconds.

a. Find h when $t = 0, 1, 2, 3,$ and 4.

b. Explain what the pairs (t, h) tell you about the height of the ball for $t = 0, 2,$ and 4.

(continued on next page)

c. Graph the pairs (t, h) over the domain of the function.

d. Is the ball moving at the same average rate (speed) between $t = 0$ and $t = 1$ as between $t = 2$ and $t = 3$? Justify your answer.

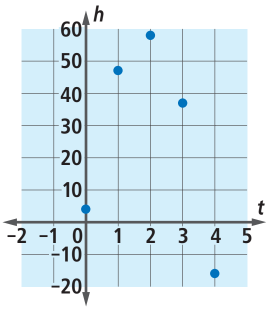
Solution

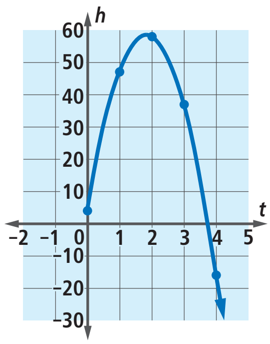
a. Use the table feature on your graphing utility or substitute by hand.

t (sec)	h (ft)
0	4
1	47
2	58
3	37
4	-16

b. Each pair (t, h) gives the height h of the ball after t seconds. The pair (0, 4) means that at 0 seconds, the time of release, the ball is 4 feet above the ground. The pair (2, 58) means the ball is 58 feet high after 2 seconds. The pair (4, -16) means that after 4 seconds, the ball is 16 feet below ground level. Unless the ground is not level, it has already hit the ground.

c. The points in Part a are plotted at left below. The points do not tell much about the shape of the graph. More points are needed to show the parabola. By calculating h for other values of t, or by using a graphing utility, you can obtain a graph similar to the one at the right below. The graph is not a complete parabola because the domain of the function is $\{t | t \geq 0\}$.





d. The average rate of change between two times is the change in height divided by the change in time. This is the slope of the line through the corresponding points on the graph.

The average rate of change between $t = 0$ and $t = 1$ is $\frac{47-4}{1-0} = 43 \frac{\text{ft}}{\text{second}}$. The average rate of change between $t = 2$ and $t = 3$ is $\frac{37-58}{3-2} = -21 \frac{\text{ft}}{\text{second}}$. (The ball is moving downward on this interval.) The rates are different, meaning the ball travels at different speeds during its flight.

By the Parabola Congruence Theorem, you know that the graph of $h = -16t^2 + 59t + 4$ is a translation image of the graph of $y = -16t^2$.

The equation in Example 2 is a special case of the following general formula that Newton developed for the height h of an object at time t seconds with an initial upward velocity v_0 and initial height h_0.

$$h = -\frac{1}{2}gt^2 + v_0t + h_0$$

In Example 2, $v_0 = 59\frac{\text{ft}}{\text{sec}}$, the height $h_0 = 4$ ft, and g is a constant denoting *acceleration due to gravity*. Recall that *velocity* involves units like miles per hour, feet per second, or meters per second. Acceleration measures how fast the velocity changes. This "rate of change of a rate of change" involves units like feet per second per second (which is usually abbreviated $\frac{\text{ft}}{\text{sec}^2}$). The acceleration due to gravity varies depending on how close the object is to the center of a massive object. Ignoring the effects of air resistance, near the surface of Earth,

$$g \approx 32\frac{\text{ft}}{\text{sec}^2}, \qquad \text{or} \qquad g \approx 9.8\frac{\text{m}}{\text{sec}^2}.$$

Two common situations are important to note. First, if an object is dropped, not thrown or pushed, its initial velocity $v_0 = 0$. Second, if an object starts at ground level, its initial height $h_0 = 0$.

STOP **QY2**

QY2

An object's height is modeled using the equation $h = -16t^2 + 24t + 4$. What is the initial velocity? (Do not forget the units.) From what height is it thrown?

Activity

MATERIALS stopwatch, meter stick, tape, rubber ball

Work with a partner to apply Newton's formula for free-falling objects.

Step 1 Copy the table below to record your data.

	Initial height h_0 (m)	Elapsed Time Trial 1 (sec)	Elapsed Time Trial 2 (sec)	Elapsed Time Trial 3 (sec)	Elapsed Time Average t (sec)
Partner 1	?	?	?	?	?
Partner 2	?	?	?	?	?

Step 2 Choose one partner to be the tosser and the other to be the measurer. The tosser chooses a comfortable height from which to toss the ball upward. The measurer records this height and marks it on the meter stick with tape so the tosser can try to consistently release the ball at the same height.

Step 3 The tosser throws the ball upward three times in succession from the height determined in Step 2. With the stopwatch, the measurer records the elapsed time, in seconds, from the initial release of the ball to when it first hits the ground.

Step 4 Reverse roles with your partner and repeat Steps 2 and 3.

Step 5 Calculate and record average times for each partner's tosses.

(continued on next page)

Step 6 Use Newton's formula, $h = -\frac{1}{2}gt^2 + v_0t + h_0$ to calculate the initial upward velocity v_0 for each partner's average toss. (Hint: When did $h = 0$?) Then write an equation to describe each partner's average toss.

Step 7 The ball reaches it maximum height in a little less than half the time it takes the ball to hit the ground. Use your formula to estimate the maximum height of your average toss.

Caution! The equation $h = -\frac{1}{2}gt^2 + v_0t + h_0$ models the height h of the object off the ground at time t. It *does not* describe the path of the object. However, Galileo showed that the actual path of an object thrown at any angle except straight up or straight down is almost parabolic, like the path of water on the second page of the chapter, and an equation for its path is a quadratic equation.

Questions

COVERING THE IDEAS

1. Write the standard form for the equation of a parabola with a vertical line of symmetry.

In 2 and 3, rewrite the equation in standard form.

2. $y = (x - 3)^2$

3. $y = -3(x + 4)^2 - 5$

4. **True or False** For any values of a, b, and c, the graph of $y = ax^2 + bx + c$ is congruent to the graph of $y = ax^2$.

In 5–7, use the equation $h = -\frac{1}{2}gt^2 + v_0t + h_0$ for the height of a body in free fall.

5. Give the meaning of each variable.

   a. h b. g c. t d. v_0 e. h_0

6. What value of g should you use if v_0 is measured in $\frac{\text{ft}}{\text{sec}}$?
7. What is the value of v_0 when an object is dropped?

In 8–11, refer to the graph in Example 2.

8. About how high is the ball after 1.5 seconds?
9. When the ball hits the ground, what is the value of h?
10. At what times will the ball be 20 feet above the ground?
11. What is the average rate of change of the ball's height between 1 second and 3 seconds?

12. Suppose a person throws a ball upward at a velocity of 16 $\frac{\text{m}}{\text{sec}}$ from the top of a 20-meter tall building.

   a. Write an equation to describe the height of the ball above the ground after t seconds.

   b. How high is the ball after 0.75 second?

   c. Use a graph to estimate the ball's maximum height.

   d. After 6 seconds, is the ball above or below ground level? Justify your answer.

APPLYING THE MATHEMATICS

13. Sketch $y = -x^2 + 4x + 6$ for $-2 \le x \le 6$. On your sketch of the graph, label the vertex and the x- and y-intercepts with approximate values.

14. Consider the function f defined by the equation $f(x) = x^2 + 3x - 10$.

   a. Sketch a graph of the function.

   b. Write an equation for the line of symmetry of the parabola.

   c. Estimate the coordinates of the lowest point on the parabola.

In 15 and 16, because the object is dropped, not thrown, its initial velocity is 0.

15. Suppose a penny is dropped from the top of Taipei 101, which in 2004 surpassed the Twin Petronas Towers in Malaysia as the world's tallest building. The roof of Taipei 101 is 1,474 feet above ground.

   a. Write an equation for the penny's height as a function of time.

   b. Graph your equation from Part a over an appropriate domain.

   c. Estimate how much time it would take the penny to fall to the ground.

   d. When the penny falls through the atmosphere, air resistance actually limits its velocity to a maximum of about 94 feet per second. If the penny traveled at a constant rate of 94 feet per second after 2.9 seconds, how much longer would it take to reach the ground?

Taipei 101 in Taipei, Taiwan

16. In an article about education now often circulated as a joke, the late Dr. Alexander Calandra suggested one way to measure the height of a building with a barometer: drop the barometer from the top of the building and time its fall.

   a. Set up an equation for the barometer's height as a function of time, using h_0 for the initial height of the building.

   b. Suppose it takes 3.9 seconds for the barometer to hit the ground. Substitute values into the equation you wrote in Part a and solve for h_0.

17. Find an equation in standard form for the image of the graph of $y = -\frac{1}{4}x^2$ under the translation $T_{4,2}$.

REVIEW

In 18 and 19, two equations are given.

a. Graph both equations on the same set of axes.

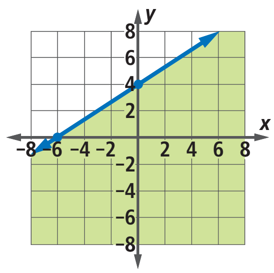
b. Describe how the graphs of the two equations are related. (Lesson 6-3)

18. $y = x^2$ and $y = (x + 3)^2 + 4$

19. $y = |x|$ and $y - 5 = |x - 2|$

20. A gallon of paint can cover an area of 450 square feet. Find the diameter of the largest circle that can be covered with a gallon of paint. (Lesson 6-2)

21. Write an inequality to describe the shaded region of the graph at the right. (Lessons 5-7, 3-4)



22. Solve the system $\begin{cases} A + B + C = 12 \\ 4A - 4B + 2C = -16 \\ 3A + 3B - C = 4 \end{cases}$. (Lesson 5-4)

In 23–25, find *n*. (Previous Course)

23. $x^2 \cdot x^3 = x^n$

24. $a^n \cdot a^{16} = a^{64}$

25. $\frac{p^8}{p^2} = p^n$

EXPLORATION

26. How do the values of a, b, and c affect the graph of $y = ax^2 + bx + c$? Here are two suggested methods for investigating:

Method 1 Use sliders on a DGS or CAS to adjust one coefficient, a, b, or c, at a time.

Method 2
a. Start with $a = 1$ and $b = 6$. Then adjust c and record how the graph changes.
b. Set $a = 1$ and $c = 4$, then adjust b and note the changes in the graph.
c. Set $b = 6$ and $c = 4$, then adjust a and note the changes.

Is the transformation (motion) of the graph simple (like a translation or rotation) for each change of a, b, and c, or is it a compound motion? Which coefficients, if any, affect the graph's size as well as its position?

QY ANSWERS

1. 0

2. 24 ft/sec; 4 ft