Lesson

6-3 The Graph-Translation Theorem

Vocabulary

corollary
vertex form
axis of symmetry
minimum, maximum

BIG IDEA If you know an equation for a graph, then you can easily find an equation for any translation image of the graph.

You can quickly graph functions whose graphs are translation images of functions with which you are already familiar.

Mental Math

Let x = 3. Find

a. $x^2 + 7$.

b. $(x + 7)^2$.

c. $7x^2$.

d. $(7x)^2$.

Activity

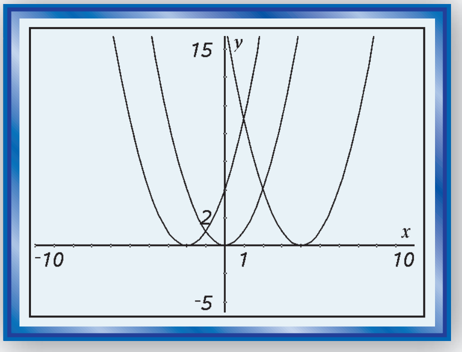
MATERIALS graphing utility

Work with a partner.

Step 1 Graph each group of equations below on the same axes. Print or sketch each group and label the individual parabolas.

Group A	Group B	Group C
$f1(x) = x^2$	$f1(x) = x^2$	$f1(x) = x^2$
$f2(x) = (x - 4)^2$	$f2(x) = x^2 - 5$	$f2(x) = (x - 3)^2 + 2$
$f3(x) = (x + 2)^2$	$f3(x) = x^2 + 3$	$f3(x) = (x + 4)^2 - 1$

Step 2 For each group, describe the translations that map the graph of $f(x) = x^2$ onto the graphs of the other two equations.



Step 3 Without graphing, describe the graph of each equation below as a translation image of the graph of $y = x^2$.

a. $y = (x + 5)^2$

b. $y = x^2 - 1$

c. $y = (x + 2)^2 - 5$

Step 4 Make some conjectures. For any real numbers h and k, what translation maps $y = x^2$ onto the graph of

a. $y = (x - h)^2$? b. $y = x^2 + k$? c. $y = (x - h)^2 + k$?

Step 5 Test your conjectures from Step 4 for some other positive values of h and k.

The Graph-Translation Theorem

Your sketch of Group A in the Activity should show that when x is replaced by $(x - 4)$, the preimage is translated 4 units to the right, and when x is replaced by $(x + 2)$, the preimage is translated 2 units to the left. In general, replacing x with $x - h$ in a mathematical sentence translates its graph h units horizontally.

Similarly, replacing y with $y - k$ in a sentence translates its graph k units vertically. For example, your sketch of Group B should show that the graph of $y = x^2 + 3$ is 3 units above the graph of $y = x^2$. Note that you can rewrite this equation as $y - 3 = x^2$, so replacing y with $y - 3$ in the equation for a function translates its graph 3 units up.

Recall that the translation $T_{h,k}$ creates an image of a figure h units to the right and k units up from its preimage. The graph of $y - 2 = (x - 3)^2$ is the translation image of the graph $y = x^2$ under $T_{3,2}$. The results of the Activity are summarized in the Graph-Translation Theorem.

Graph-Translation Theorem

In a relation described by a sentence in x and y, the following two processes yield the same graph:

1. replacing x by $x - h$ and y by $y - k$;
2. applying the translation $T_{h,k}$ to the graph of the original relation.

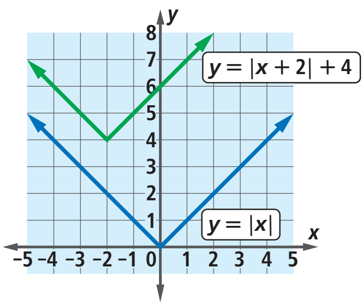
The Graph-Translation Theorem applies to all relations that can be described by a sentence in x and y.

Example 1

Find an equation for the image of the graph of $y = |x|$ under the translation $T_{-2,4}$.

Solution Applying $T_{-2,4}$ is equivalent to replacing x with $x - (-2)$, or $x + 2$, and y with $y - 4$ in the equation for the preimage. An equation for the image is $y - 4 = |x - (-2)|$, or $y = |x + 2| + 4$.

Check $T_{-2,4}$ is the translation that slides a figure 2 units left and 4 units up. Graph $y = |x|$ and $y = |x + 2| + 4$ on the same set of axes. As shown at the right, the graph of the second equation is the image of the graph of the first equation under a translation 2 units to the left and 4 units up. It checks.



Using the Graph-Translation Theorem to Graph Parabolas

Recall from Chapter 2 that the graph of $y = ax^2$ is a parabola. If we replace x with $x - h$ and y with $y - k$ in the equation $y = ax^2$, we obtain $y - k = a(x - h)^2$. Because a figure is congruent to its translation image, the graph of this equation is also a parabola.

This argument proves the following *corollary* to the Graph-Translation Theorem. A **corollary** is a theorem that follows immediately from another theorem.

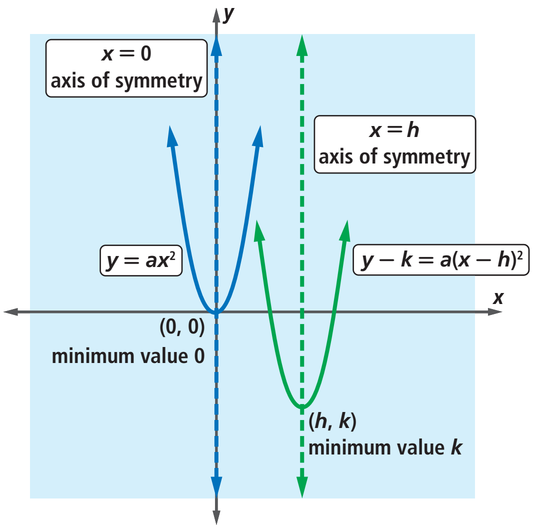
Parabola-Translation Theorem

The image of the parabola with equation $y = ax^2$ under the translation $T_{h,k}$ is the parabola with the equation

$$y - k = a(x - h)^2 \text{ or}$$
$$y = a(x - h)^2 + k.$$

The Graph-Translation Theorem and the Parabola-Translation Theorem can help you identify characteristics of a parabola by looking at its equation. As you read about these characteristics below, look at the graphs of $y = ax^2$ and $y - k = a(x - h)^2$ at the right.

- *Vertex* You know that (0, 0) is the vertex of the parabola $y = ax^2$. Under $T_{h,k}$, the translation image of (0, 0) is $T_{h,k}(0, 0) = (0 + h, 0 + k) = (h, k)$. So, the vertex of the parabola with equation $y - k = a(x - h)^2$ is (h, k). For this reason, the equation $y - k = a(x - h)^2$ is called the **vertex form** of an equation of a parabola.



- *Axis of symmetry* The parabola with equation $y = ax^2$ is reflection-symmetric to the y-axis, which has equation $x = 0$. Since the parabola is translated h units to the right under $T_{h,k}$, the line with equation $x = h$ is the symmetry line or **axis of symmetry** of the parabola with equation $y - k = a(x - h)^2$.
- *Maximum or minimum y-value* If $a > 0$, then the parabola with equation $y - k = a(x - h)^2$ opens up and the y-coordinate of the vertex is the **minimum** y-value. The graphs at the right picture the functions when $a > 0$. If $a < 0$, then the parabola opens down and the y-coordinate of the vertex is the **maximum** y-value.

Knowing these facts helps you to quickly sketch parabolas by hand and to better understand what you see when you use a graphing utility.

GUIDED

Example 2

a. State the coordinates of the vertex of the parabola with equation $y - 5 = -2(x + 3)^2$.

b. Write an equation for the axis of symmetry of the parabola.

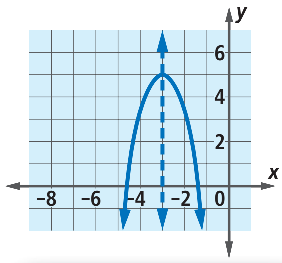
c. Sketch a graph of the equation by hand.

Solution

a. The equation $y - 5 = -2(x + 3)^2$ results from replacing x with ___?___ and y with ___?___ in $y = -2x^2$. So its graph is the image of $y = -2x^2$ under the translation ___?___. The graph is a parabola with vertex (___?___, ___?___).

b. Since the axis of symmetry of $y = -2x^2$ has equation ___?___, the axis of symmetry of this parabola is the line with equation $x =$ ___?___.

c. Because the graph of $y = -2x^2$ opens down, so does the graph of $y - 5 = -2(x + 3)^2$. Find a point on the graph other than the vertex. For example, let $x = -2$. Then $y - 5 = -2($ ___?___ $+ 3)^2$, so $y =$ ___?___ $+ 5$ $=$ ___?___, and (–2, ___?___) is a point on the graph. Sketch a parabola with vertex ___?___, opening downward through the point (–2, ___?___) and symmetric to the line $x =$ ___?___.



STOP QY

Finding Equations for Parabolas

You can apply the Graph-Translation Theorem to a known parabola to find an equation for its image under a given translation.

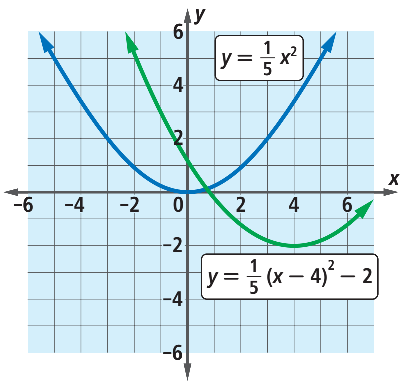
QY

Describe the graph of $y - 15 = (x + 40)^2$.

Example 3

Consider the parabolas at the right. The one that passes through the origin has equation $y = \frac{1}{5}x^2$. The other is its image under a translation. Find an equation for the image.

Solution The translation image appears to be 4 units to the right and 2 units down from the preimage. So the translation is $T_{4,-2}$. Applying $T_{4,-2}$ is equivalent to replacing x with $x - 4$ and y with $y - (-2) = y + 2$ in the equation for the preimage. An equation for the image is $y + 2 = \frac{1}{5}(x - 4)^2$.



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Check Use a graphing utility. Because $y + 2 = \frac{1}{5}(x - 4)^2$ is equivalent to $y = \frac{1}{5}(x - 4)^2 - 2$, plot $y = \frac{1}{5}x^2$ and $y = \frac{1}{5}(x - 4)^2 - 2$ in the same window. You should see that the graph of the second equation is the image of the graph of the first under $T_{4,-2}$.

You will see more applications of the Graph-Translation Theorem in later chapters.

Questions

COVERING THE IDEAS

In 1 and 2, tell how the graphs of the two equations are related.

1. $y_1 = x^2$ and $y_2 = (x - 5)^2$
2. $y_1 = x^2$ and $y_2 + 6 = x^2$
3. a. What is the image of (x, y) under $T_{7,0}$?
   b. Under $T_{7,0}$, what is an equation for the image of the graph of $y = x^2$?
4. On the same axes, sketch $y = |x|$ and $y + 1 = |x - 4|$.
   a. Describe how the two graphs are related.
   b. What translation maps the first onto the second?
5. Suppose the translation $T_{4,-7}$ is applied to the parabola with equation $y = \frac{7}{5}x^2$. Find an equation for the image.
6. **Fill in the Blanks** The graph of $y - k = a(x - h)^2$ is __?__ units above and __?__ units to the right of the graph of $y = ax^2$.
7. **True or False** For all values of h and k, the graphs of $y = ax^2$ and $y = a(x - h)^2 + k$ are congruent.
8. a. What is the vertex of the parabola with equation $y - 7 = -3(x + 5)^2$?
   b. What is the vertex of the parabola with equation $y - k = a(x - h)^2$?
9. a. What is an equation of the axis of symmetry of the parabola with equation $y - 7 = -3(x + 5)^2$
   b. What is an equation of the axis of symmetry of the parabola with equation $y - k = a(x - h)^2$?

In 10 and 11, an equation for a parabola is given.

a. Give the coordinates of the vertex of the parabola.

b. Give an equation for the axis of symmetry.

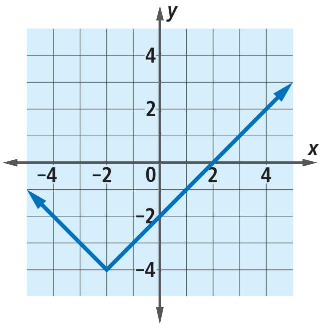
c. Tell whether the parabola opens up or opens down.

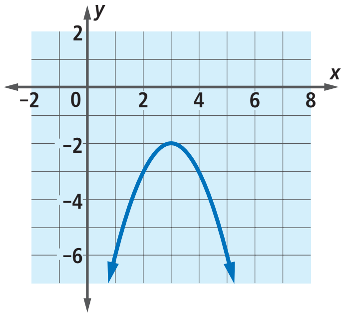
d. Sketch a graph of the equation.

10. $y + 1 = 5(x + 10)^2$
11. $y = 5 - (x - 4)^2$

Parabolic arcs can often be found in art and architecture, as in Casa Mila, located in Barcelona, Spain, shown here.

12. Find an equation for the translation image of $y = |x|$ graphed at the left below.





13. The parabola graphed at the right above is a translation image of $y = -x^2$. What is an equation for this parabola?

APPLYING THE MATHEMATICS

14. On the first page of this chapter, an equation for the path of water from a drinking fountain is given as $y = -0.58x^2 + 2.7x$, where x and y are measured in inches. This equation is roughly equivalent to $y - 3.14 = -0.58(x - 2.33)^2$. From the second equation, how high does the water reach?

Centennial Fountain was built in Chicago in 1989 and goes off every hour during the summer months.

15. Consider the graph of $y = -4x^2$. Write an equation for a translation image of the graph

a. with vertex $(0, 2)$. **b.** with vertex $(2, 0)$.

16. **a.** Solve $x^2 = 81$.

b. Solve $(x - 3)^2 = 81$.

c. How are the solutions in Parts a and b related to the Graph-Translation Theorem?

17. One solution to $x^2 + 8x + 9 = 57$ is 4. Use this information to find a solution to $(x - 5)^2 + 8(x - 5) + 9 = 57$.

18. Find the x-intercepts and y-intercept of the graph of $y = |x + 3| - 5$.

19. **Fill in the Blanks** The point-slope form of a line, $y - y_1 = m(x - x_1)$, can be thought of as the image of the line with equation __?__ under the translation $T_{h,k}$, where $h =$ __?__ and $k =$ __?__.

20. The parabola $y = 2(x - 4)^2 - 2$ is the image of the parabola $y = 2(x - 3)^2 + 5$ under the translation $T_{h,k}$. What are the values of h and k?

REVIEW

In 21 and 22, solve and check. **(Lesson 6-2)**

21. $3 \cdot |2d + 3| = 21$

22. $-8x^2 = -162$

23. The competition area for a judo contest consists of a d-meter by d-meter square surrounded by a 3-meter-wide border called the *safety area*. **(Lesson 6-1)**

a. Write an expression in standard form for the total area of the competition area.

b. The rules of judo require $8 \le d \le 10$. What are the minimum and maximum areas a judo competition area can have?

Judo emphasizes flexibility, energy, and balance, rather than brute strength.

24. A company makes two kinds of tires: model R (regular) and model S (snow). Each tire is processed on three machines, A, B, and C. To make one model R tire requires $\frac{1}{2}$ hour on machine A, 2 hours on B, and 1 hour on C. To make one model S tire requires 1 hour on A, 1 hour on B, and 4 hours on C. During the upcoming week, machine A will be available for at most 20 hours, machine B for at most 60 hours, and machine C for at most 60 hours. If the company makes a \$10 profit on each model R tire and a \$15 profit on each model S tire, how many of each tire should be made to maximize the company's profit? **(Lesson 5-9)**

25. Simplify the expression $\frac{(x^2y)^2}{y^3}$. **(Previous Course)**

EXPLORATION

26. Investigate how the Graph-Translation Theorem works with other functions. Graph $y = x^3 - 4x$ on your graphing utility. Pick values for h and k, write an equation for its image under each translation below, and graph the image. Verify that the image appears to be a translation image of $y = x^3 - 4x$. Try other values of h and k.

a. $T_{0,k}$

b. $T_{h,0}$

c. $T_{h,k}$

QY ANSWER

The graph is the image under $T_{-40,15}$ of the graph of $y = x^2$. It has vertex $(-40, 15)$, is reflection-symmetric to $x = -40$, and opens up.