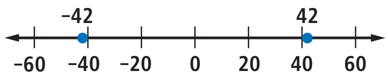
Lesson 6-2 Absolute Value, Square Roots, and Quadratic Equations

Vocabulary

absolute value
absolute-value function
square root
rational number
irrational number

BIG IDEA Geometrically, the *absolute value* of a number is its distance on a number line from 0. Algebraically, the absolute value of a number equals the nonnegative square root of its square.

The **absolute value** of a number n, written $|n|$, can be described geometrically as the distance of n from 0 on the number line. For instance, $|42| = 42$ and $|-42| = 42$. Both 42 and –42 are 42 units from zero.



Algebraically, the absolute value of a number can be defined piecewise as follows.

$$|x| = \begin{cases} x, \text{ for } x \geq 0 \\ -x, \text{ for } x < 0 \end{cases}$$

Examine the definition carefully. Because $-x$ is the opposite of x, $-x$ is positive when x is negative. For instance, $|-7.4| = -(-7.4) = 7.4$. Thus $|x|$ are $|-x|$ are never negative, and, in fact, $|x| = |-x|$.

On many graphing utilities, spreadsheets, and CAS, the absolute-value function is denoted `abs`. For example, `abs(x–3) = |x–3|`.

Mental Math

A company makes $6 dollars in revenue for every teacup it sells and $5 in revenue for every saucer it sells. How much revenue will the company make if they sell

a. 500 teacups and no saucers?

b. 400 teacups and 200 saucers?

c. 500 saucers and no teacups?

Example 1

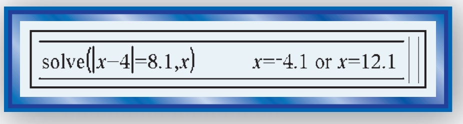
Solve for x: $|x - 4| = 8.1$.

Solution Use the algebraic definition of absolute value.

Either $x - 4 = 8.1$ or $x - 4 = -8.1$.

So, $x = 12.1$ or $x = -4.1$.

Check Use a CAS.

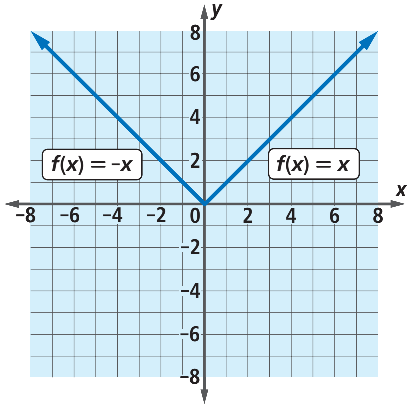


STOP QY1

QY1

Suppose $f(x) = |x - 1|$. Write a piecewise definition for f.

The Absolute-Value Function



Because every real number has exactly one absolute value, $f\colon x \to |x|$ is a function. The graph of $f(x) = |x|$ is shown at the right. When $x \geq 0$, $f(x) = x$, and the graph is a ray with slope 1 and endpoint (0, 0). This is the ray in the first quadrant. When $x \leq 0$, $f(x) = -x$, and the graph is the ray with slope −1 and endpoint (0, 0). This is the ray in the second quadrant. The graph of $f(x) = |x|$ is the union of two rays, so the graph of $f(x) = |x|$ is an angle.

This function is called the **absolute-value function**. Its domain is the set of real numbers, and its range is the set of nonnegative real numbers.

Absolute Value and Square Roots

The simplest quadratic equations are of the form $x^2 = k$. When $k \geq 0$, the solutions to $x^2 = k$ are the positive and negative **square roots** of k, namely $\sqrt{k}$ and $-\sqrt{k}$. Square roots are intimately connected to absolute value.

Activity

Consider the functions f and g with equations $f(x) = \sqrt{x^2}$ and $g(x) = |x|$.

Step 1 In each row of the table, choose a value of x satisfying the constraint. Then evaluate $f(x)$ and $g(x)$. One row is completed for you.

Step 2 Make a conjecture about the relationship between f and g.

Step 3 Graph f and g on the same axes.

Step 4 Trace and toggle between the graphs to compare $f(x)$ and $g(x)$ for several values of x. Explain the apparent relationship between the graphs of f and g.

Constraint	x	$f(x)$	$g(x)$
$x < -10$	?	?	?
$-10 \leq x \leq -1$	?	?	?
$-1 < x < 0$	?	?	?
$x = 0$	0	0	0
$0 < x < 1$	?	?	?
$1 \leq x \leq 10$	?	?	?
$x > 10$	?	?	?

The Activity suggests that, for all real numbers x, $\sqrt{x^2}$ is equal to $|x|$.

Absolute Value–Square Root Theorem

For all real numbers x, $\sqrt{x^2} = |x|$.

Proof Either $x > 0$, $x = 0$, or $x < 0$.

If $x > 0$, then $\sqrt{x^2} = x$, and also $|x| = x$, so $\sqrt{x^2} = |x|$.

If $x = 0$, then $\sqrt{x^2} = 0$, and also $|x| = |0| = 0$, so $\sqrt{x^2} = |x|$.

If $x < 0$, then $\sqrt{x^2} = -x$, and also $|x| = -x$, so $\sqrt{x^2} = |x|$.

Solving $ax^2 = b$

The Absolute Value–Square Root Theorem can be used to solve quadratic equations of the form $ax^2 = b$.

Example 2

Solve $x^2 = 12$.

Solution 1 Take the positive square root of each side.

$\sqrt{x^2} = \sqrt{12}$

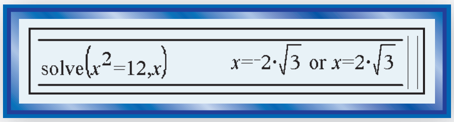
Use the Absolute Value–Square Root Theorem.

$|x| = \sqrt{12}$

So, either $x = \sqrt{12}$ or $x = -\sqrt{12}$.

Check Use your calculator to evaluate $(\sqrt{12})^2$ and $(-\sqrt{12})^2$. Each equals 12. It checks.

Solution 2 Use a CAS.

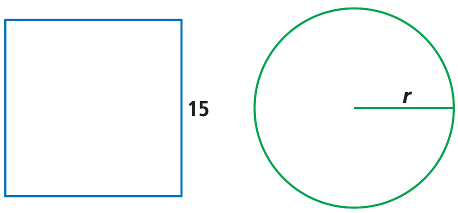


Check The solutions are shown as $-2\sqrt{3}$ and $2\sqrt{3}$, so multiply to show that $(-2\sqrt{3})^2$ and $(2\sqrt{3})^2$ both equal 12.

When $x = a$ or $x = -a$, you can write $x = \pm a$. In Example 2, $x = \pm\sqrt{12} = \pm 2\sqrt{3}$.

Example 3

A square and circle have the same area. The square has side length 15 units. Which is longer, a side of the square or the diameter of the circle?



Solution The area of the square is $15 \cdot 15 = 225$ square units.

Since we know a formula for the area of a circle in terms of its radius, let *r* be the radius of the circle.

$$\pi r^2 = 225$$

$$r^2 = \frac{225}{\pi} \quad \text{Divide by } \pi.$$

$$|r| = \sqrt{\frac{225}{\pi}} \quad \text{Take the square root of each side and use the Absolute Value–Square Root Theorem.}$$

$$r = \pm\sqrt{\frac{225}{\pi}} \quad \text{Definition of absolute value}$$

$$\approx \pm 8.46 \text{ units}$$

You can ignore the negative solution because a radius cannot be negative. The radius of the circle is approximately 8.5 units. So the diameter is about 17 units and is longer than a side of the square.

STOP QY2

QY2

Check the solution to Example 3. Is $\pi(8.5)^2 \approx 225$?

Rational and Irrational Numbers

Recall from earlier courses that a *simple fraction* is a fraction of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. And recall from Chapter 1 that a number that can be written as a simple fraction is called a **rational number**. Around 430 BCE, the Greeks proved that unless an integer is a perfect square (like 49, 625, or 10,000), its square root is an *irrational number*. An **irrational number** is a real number that cannot be written as a simple fraction. Irrational numbers, including most square roots, have infinite nonrepeating decimal expansions. The exact answers to Examples 2 and 3 are irrational numbers.

READING MATH

As used in algebra, the word *rational* comes from the word *ratio*. A *rational number* is a number that can be written as the *ratio* of two integers.

Questions

COVERING THE IDEAS

1. Evaluate without a calculator.
   a. $|17.8|$ b. $|-17.8|$ c. $-|17.8|$ d. $-|-17.8|$
2. A classmate believes $|-t| = t$ for all real numbers t. Is this correct? Explain your answer.
3. A classmate believes `abs(x) = -abs(x)` for all real numbers x. Is this correct? Why or why not?
4. Sketch a graph of f and g with equations $f(x) = |x - 4|$ and $g(x) = 8.1$, and label the coordinates of the points of intersection to verify the answer to Example 1.
5. The two numbers at a distance 90 from 0 on a number line are the solutions to what equation?

In 6 and 7, solve.

6. $|3.4 - y| = 6.5$

7. $|2n + 7| = 5$

8. Consider the function f with equation $f(x) = -|x|$.
   a. State its domain and its range.
   b. **True or False** The graph of f is piecewise linear. Justify your answer.

9. **Multiple Choice** What is the solution set to $\sqrt{x^2} = |x|$?
   A the set of all real numbers
   B the set of all nonnegative real numbers
   C the set of all positive real numbers

In 10 and 11, find all real-number solutions to the nearest thousandth.

10. $k^2 = 261$

11. $3x^2 = 2187$

12. a. Find the exact radius of a circle whose area is 150 square meters.
    b. Estimate the answer to Part a to the nearest thousandth.

13. A circle has the same area as a square with side length 8. What is the radius of the circle to the nearest hundredth?

14. A square has the same area as a circle with radius 9. What is the length of a side of the square to the nearest hundredth?

In 15–20, tell whether the number is rational or irrational. If it is rational, write the number as an integer or a simple fraction.

15. $\sqrt{8}$

16. $\sqrt{100} - 2$

17. $\sqrt{36}$

18. $\frac{0.13}{713}$

19. $\frac{2}{\sqrt{2}}$

20. π

APPLYING THE MATHEMATICS

21. The formula $e = |p - I|$ gives the allowable margin of error e for a given measurement p when I is the ideal measurement. A certain soccer ball manufacturer aims for a weight of 442.5 g with an acceptable value of e being no more than 1.5 g.
    a. Use absolute value to write a mathematical sentence for the allowable margin of error for soccer ball weights p.
    b. What is the most a soccer ball from this manufacturer should weigh?

22. a. Graph $f(x) = -2\sqrt{(x + 3)^2}$ and $g(x) = -2|x + 3|$ on the same set of axes in a standard window.
    b. How do the two graphs appear to be related?

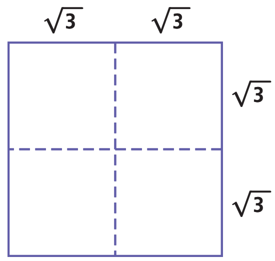
23. The directions on a brand-name pizza box read, "Spread dough to edges of a round pizza pan or onto a 10" by 14" rectangular baking sheet." How big a circular pizza could you make with this dough, assuming it is spread the same thickness as for the rectangular pizza?

24. Graph $f(x) = |x + 2|$ and $h(x) = |x| + 2$ on the same set of axes in a standard window.

   a. According to the graph, for which values of x does $f(x) = g(x)$?

   b. Describe the set of numbers for which $f(x) \neq g(x)$.

25. Use the drawing at the right to explain why $2\sqrt{3} = \sqrt{12}$.



REVIEW

In 26 and 27, multiply and simplify. (Lesson 6-1)

26. $(x + 3y)(x - 2y)$

27. $(8 + x)(8 - x)$

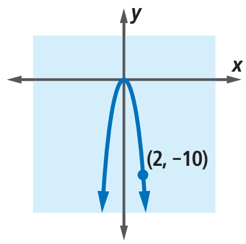
28. Consider the line with equation $y = \frac{4}{3}x + 3$. Find an equation for the image of this line under the translation $T_{3,1}$. (Lesson 4-10)

29. a. Graph the first eight terms of the sequence defined recursively by $\begin{cases} v_1 = 1 \\ v_n = v_{n-1} + n, \text{ for integers } n \geq 2 \end{cases}$.

   b. Rewrite the second line of the formula in Part a if v_n represents the previous term of the sequence.

   (Lessons 3-7, 3-6)

30. A graph of $y = kx^2$ is shown at the right. Find the value of k. (Lesson 2-5)



EXPLORATION

31. One way to estimate $\sqrt{k}$ without using a square root command on a calculator or computer uses the following sequence:

$$\begin{cases} a_1 = \text{initial guess at the root} \\ a_n = \frac{1}{2}\left(a_{n-1} + \frac{k}{a_{n-1}}\right), \text{ for integers } n \geq 2. \end{cases}$$

   a. Let $k = 5$. Give a rational number approximation for $\sqrt{5}$ and use that number as a_1. Then find a_2, a_3, a_4, and a_5. Use a calculator to check the difference between a_5 and $\sqrt{5}$.

   b. Continue to generate terms of the sequence until you are within 0.0001 of $\sqrt{5}$.

   c. Use the sequence to estimate the positive square root of 40 to the nearest millionth.

QY ANSWERS

1. $f(x) = |x - 1| = \begin{cases} x - 1, & \text{for } x \geq 1 \\ -x + 1, & \text{for } x < 1 \end{cases}$

2. $\pi(8.5)^2 \approx 226.98$, close enough given that 8.5 is an approximation. (In fact, $\pi(8.46)^2 \approx 224.85$, much closer to 225.)