Lesson

6-1

Quadratic Expressions, Rectangles, and Squares

▶ **BIG IDEA**   Some quadratic expressions arise from problems involving the areas of rectangles.

Recall that the degree of a monomial is the sum of the exponents of the variables in the monomial, and the degree of a polynomial is the highest degree among its monomial terms. For instance, the monomials $20x^5$ and $\frac{1}{3}ab^4$ have degree 5, while the polynomial $x^5 - x^6$ has degree 6. The word *quadratic* in today's mathematics refers to expressions, equations, and functions that involve sums of constants and first and second powers of variables and no higher powers. That is, they are of degree 2. Specifically, if a, b, and c are real numbers, $a \neq 0$, and x is a variable,

$ax^2 + bx + c$ is the general **quadratic expression** in x,

$ax^2 + bx + c = 0$ is the general **quadratic equation** in x, and

$f: x \rightarrow ax^2 + bx + c$ is the general **quadratic function** in x.

We call $ax^2 + bx + c$ the **standard form of a quadratic**. In standard form, the powers of the variable are in decreasing order. Some quadratic expressions, equations, and functions are not in standard form, but they can be rewritten in standard form.

🛑 **QY1**

There can also be quadratics in two or more variables. The general quadratic expression in two variables is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Quadratic equations and quadratic functions in two variables are discussed in Chapter 12.

The simplest quadratic expression x^2 is the product of the simplest linear expressions x and x. More generally, the product of any two linear expressions $ax + b$ and $cx + d$ is a quadratic expression for any real numbers a, b, c, and d, provided a and c are not zero. Because all area formulas involve the product of two lengths, they all involve quadratic expressions.

Mental Math

A 6' by 3.5' mural is surrounded by a 1.5'–wide border.

a. What is the area of the mural?

b. What is the area of the mural plus the border?

c. What is the area of just the border?

d. The border is narrowed to 1 foot wide. Now what is the area of the mural and border?

▶ **QY1**

Write $3x - (5 - 2x^2)$ in standard form.

Quadratic Expressions from Rectangles

Example 1

Hector and Francisca are remodeling their kitchen. They purchase a 6-foot by 2–foot pantry door, and are looking at different widths of molding to trim the door frame. If the trim is w inches wide, write the total area of the door and trim in standard form.

Solution Draw a picture. The door is surrounded by trim on 3 sides. The door with trim occupies a rectangle with length $72 + w$ inches and width $24 + 2w$ inches. The area of this rectangle is $(72 + w)(24 + 2w)$ square inches.

Use the Distributive Property to multiply $(72 + w)(24 + 2w)$. Think of $(72 + w)$ as a single number.

$$(72 + w)(24 + 2w) = (72 + w) \cdot 24 + (72 + w) \cdot 2w$$

Now apply the Distributive Property twice more.

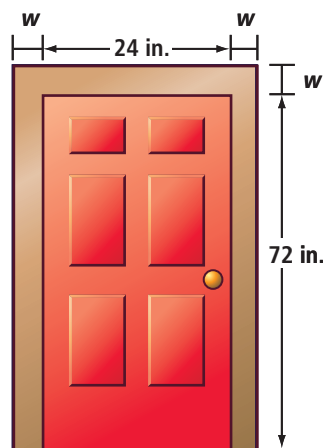$$\begin{aligned} &= 72 \cdot 24 + w \cdot 24 + 72 \cdot 2w + w \cdot 2w \\ &= 1728 + 24w + 144w + 2w^2 \qquad \text{Arithmetic} \\ &= 1728 + 168w + 2w^2 \qquad \text{Combine like terms.} \end{aligned}$$

In standard form, the total area of the door and the trim is $2w^2 + 168w + 1728$ square inches.

Check Use a CAS to expand the expression.

$$\text{expand}\big((72+w) \cdot (24+2 \cdot w)\big)$$
$$2 \cdot w^2 + 168 \cdot w + 1728$$

STOP QY2

▶ **QY2**

The expression $2w^2 + 168w + 1728$ is in the form $ax^2 + bx + c$. What are $a, b, c,$ and x?

Quadratic Expressions from Squares

The expression $x + y$ is an example of a *binomial*. In general, a **binomial** is an expression with two terms. The square of a binomial can be thought of as the area of a square whose side length is the binomial.
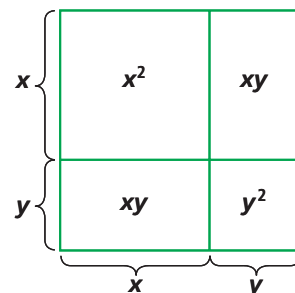
Example 2

Write the area of the square with sides of length $x + y$ in standard form.

Solution 1 Draw a picture of the square. Notice that its area is the sum of four smaller areas: a square of area x^2, two rectangles, each with area xy, and a square with area y^2. So, the area of the original square is $x^2 + 2xy + y^2$.

Solution 2 The area of a square with side $x + y$ is $(x + y)^2$.
Rewrite $(x + y)^2$.

$$
\begin{aligned}
(x + y)^2 &= (x + y)(x + y) && \text{Definition of second power} \\
&= (x + y)x + (x + y)y && \text{Distributive Property} \\
&= x^2 + yx + xy + y^2 && \text{Distributive Property} \\
&= x^2 + 2xy + y^2 && \text{Commutative Property of} \\
& && \text{Multiplication and} \\
& && \text{Distributive Property}
\end{aligned}
$$

The area of the square is $x^2 + 2xy + y^2$.

When a linear expression is multiplied by itself, or squared, the result is a quadratic expression. In Example 2, the linear expression $x + y$ is squared. You can also say it is "taken to the 2nd power." Writing this power as a quadratic expression is called *expanding* the power. Squares of binomials occur so often that their expansions are identified as a theorem.

Binomial Square Theorem

For all real numbers x and y,
$$(x + y)^2 = x^2 + 2xy + y^2 \text{ and}$$
$$(x - y)^2 = x^2 - 2xy + y^2.$$

Solution 2 of Example 2 provides the proof of the first part of this theorem. You are asked to complete the second part of this proof in Question 13. The Binomial Square Theorem is so useful that you will want to be able to apply it automatically.

🛑 **QY3**

▶ **QY3**

Use the Binomial Square Theorem to expand $(3x - y)^2$.

GUIDED

Example 3

A city wants to cover the seating area around a circular fountain with mosaic tiles. The radius of the fountain, including s feet for seating, is 15 feet.

a. Write a quadratic expression in standard form for the area of the fountain, not including the seating area.

b. How many square feet are in the seating area in terms of s?

Solution

a. Draw a picture. The radius of the fountain without seating is ___?___ ft. So, the fountain area without seating is $\pi ($___?___$)^2$ ft^2. To expand $($___?___ $-$ ___?___$)^2$, use the Binomial Square Theorem with $x = $ ___?___ and $y = s$.
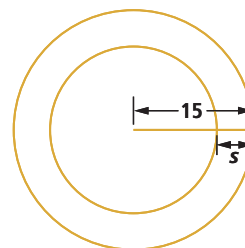
$$(\underline{\;?\;} - s)^2 = \underline{\;?\;}^2 - 2 \cdot \underline{\;?\;} \cdot s + s^2$$
$$= s^2 - \underline{\;?\;} s + \underline{\;?\;}$$

So the area of the fountain not including seating is
$\pi(\underline{\;?\;})^2$ or $\pi \underline{\;?\;} - 30\pi s + 225\pi$ ft^2.

b. The seating area is the total area minus the fountain area without seating.

$$15^2\pi - (\underline{\;?\;})^2\pi = 225\pi - (\underline{\;?\;} - \underline{\;?\;} + \underline{\;?\;})\pi$$
$$= \underline{\;?\;} - \underline{\;?\;}$$

So the area of the seating area is $\underline{\;?\;}$ ft^2.

Questions

COVERING THE IDEAS

1. **Multiple Choice** Which is not a quadratic equation? Explain your answer.

   A $y = \frac{x^2}{6}$ **B** $y = 2(x - 4)$

   C $\frac{x^2}{4} - \frac{y^2}{9} = 25$ **D** $y = (x + 3)(2x - 5)$

2. Is $x^2 + \sqrt{7}$ a quadratic expression? Explain your answer.

3. Name two geometric figures for which quadratic expressions describe their area.

4. A door is 7 feet high and 30 inches wide, with trim of w inches wide on three sides of the door frame. Write the total area (in square inches) of the door and trim in standard form.

In 5–8, the product of two linear expressions is given.
   a. Rewrite the product as a single polynomial.
   b. Check your results using the expand command on a CAS.

5. $(3x + 5y)(2x + 7y)$ 6. $(x - 3)(x + 2)$

7. $(1 - 2y)(2 - 3y)$ 8. $(6 + b)(2 - b)$

In 9–11, expand the square of the binomial.

9. $(10 + 3)^2$ 10. $(d - 6)^2$ 11. $(p + w)^2$

12. Draw a geometric diagram of the expansion of $(x + 5)^2$.

13. Prove the second part of the Binomial Square Theorem.

In 14 and 15, rewrite the expression in the form $ax^2 + bx + c$.

14. $\left(3t - \frac{1}{3}\right)^2$ 15. $(1 - p)^2$

This doorway in Dublin, Ireland, has very ornate trim.

16. Refer to the quadratic expression in Example 1. Graph
$y = (72 + x)(24 + 2x)$ and $y = 2x^2 + 168x + 1728$ in the window
$\{x \mid -160 \le x \le 160\}$, $\{y \mid -2250 \le y \le 5750\}$.

   a. Trace the graph and toggle between the two graphs for at
least three different values of x. What do you notice about the
ordered pairs when you switch between the graphs?

   b. Look at a table of values for the two functions. Does the table
support your observation from Part a?

   c. Based on your results in Parts a and b, what do you conclude
about the two equations you graphed?

17. Suppose a rectangular swimming pool with dimensions of
100 feet by 12 feet is surrounded by a walkway of width w.

   a. Write a quadratic expression in standard form that
gives the area of the pool and walkway together.

   b. Write an expression that gives the area of the
walkway only.

APPLYING THE MATHEMATICS

**In 18 and 19, rewrite the expression in the form $ax^2 + bxy + cy^2$.**

18. $(2a + 3b)^2$ 19. $\left(2t - \frac{k}{3}\right)^2$

20. Refer to Example 3. If $s = 3$ feet and 12 mosaic tiles cover one
square foot, how many tiles would be needed to cover the
seating area around the fountain?

21. Certain ceramic tiles are 4 inches by 8 inches and
are separated by grout seams that are x inches wide.

   a. Write a quadratic expression in standard form for
the area covered by each tile and its share of the
grout. (The grout in each seam is shared by two
tiles, so each tile's share is only half the grout in
each seam.)

   b. If the grout seams are $\frac{1}{2}$ inch wide, approximately
how many tiles will it take to cover a 5-foot by 10-foot wall?

   c. What percent of the wall in Part b is grout?

   d. If the grout seams are $\frac{3}{4}$-inch wide, what percent of the wall
will be grout?
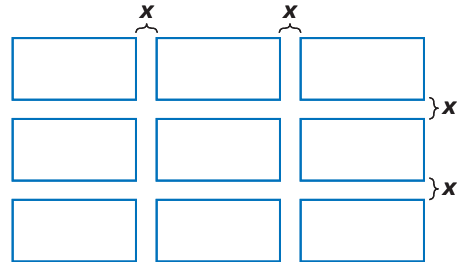
22. a. Expand $(x - y)^2 - (x + y)^2$ on a CAS.

   b. Verify the CAS solution by using the
Binomial Square Theorem.

In 23 and 24, find h so that the given equation is true.

23. $x^2 + 20x + 100 = (x + h)^2$ 24. $x^2 - hx + h = (x - \sqrt{h})^2$

REVIEW

In 25 and 26, use the following data about the United States National Parks. (Lessons 3–5, 2–4)

Year	Number of Recreation Visits (millions)	Federal Appropriations (billions of $)
1999	287.1	2.030
2000	285.9	2.112
2001	279.9	2.568
2002	277.3	2.654
2003	266.1	2.546

Source: National Park Service

25. Find the rate of change from 1999 to 2003 for each of the following:
    a. number of recreation visits   b. federal appropriations

26. a. Find an equation for the line of best fit describing the number of recreation visits as a function of the year using only the years 1999, 2001, and 2003. Let x be the number of years after 1999.
    b. How well does the line of best fit predict the value for 2002?

27. a. Draw $\triangle ABC$ with vertices $A = (0, 0)$, $B = (1, 1)$, and $C = (2, 4)$.
    b. Draw $\triangle A'B'C'$, its image under the transformation $(x, y) \rightarrow (x - 5, y + 2)$.
    c. Describe the effect of this transformation on $\triangle ABC$. (Lesson 4-10)

28. If m varies inversely as t^2, and $m = 14$ when $t = 2.5$, find the value of m when $t = 7$. (Lesson 2–2)

EXPLORATION

29. Doorway trim comes in various widths.
    a. Find out the prices of at least three different widths of doorway trim.
    b. Suppose you want to frame a 7-foot by 3-foot door. Find the area of the door moldings for each of the three different sizes you have found.
    c. How much will it cost to frame the door with each size of molding?
    d. Does the cost in Part b vary directly as the area of the molding? If so, how?

QY ANSWERS

1. $2x^2 + 3x - 5$

2. $a = 2, b = 168,$ $c = 1728,$ and $x = w$

3. $(3x)^2 - 2(3x)y + y^2 =$ $9x^2 - 6xy + y^2$