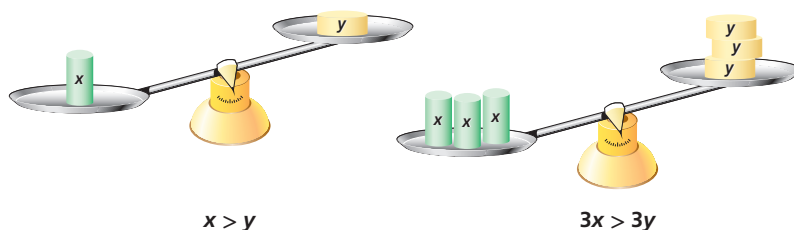# Lesson 3-6
Inequalities and Multiplication

▶ **BIG IDEA** Multiplying each side of an inequality by a positive number keeps the direction of the inequality; multiplying each side by a negative number reverses the direction of the inequality.

$x > y$

$3x > 3y$

Mental Math

Estimate to the nearest dollar.

a. 20% tip on a $49.56 bill

b. 20% tip on a $149.56 bill

c. 20% tip on a $249.56 bill

Graphs and Inequalities on a Number Line

An **inequality** is a mathematical sentence with one of the verbs < ("is less than"), > ("is greater than"), ≤ ("is less than or equal to"), or ≥ ("is greater than or equal to").

Even though they look similar, equations and inequalities are different in important ways. Consider $x = 3$, $x > 3$, and $x \geq 3$. The equation $x = 3$ has just one solution, while the inequalities $x > 3$ and $x \geq 3$ have infinitely many solutions. Their graphs at the right show these differences.

In the graph of $x > 3$ on a number line, the 3 is marked with an open circle because 3 does not make the sentence $x > 3$ true. The number 3 is the **boundary point** between the values that satisfy $x > 3$ and those that do not. Numbers just a little larger than 3 such as 3.01 and $3\frac{34}{10,000}$ are solutions, as are larger numbers like 1 million. The graph of $x \geq 3$ does include 3 because the sentence $3 \geq 3$ is true. Another way to write $x \geq 3$ is to use set-builder notation. It is written as $\{x: x \geq 3\}$ and is read as "the set of all x such that x is greater than or equal to 3."

When solving real-world problems, you must often decide what domain makes sense for the situation. Inequalities are common in the world. For example, let $x =$ a Fahrenheit temperature at which water is in solid form (ice). Then, since water freezes below 32°F, $x < 32$, as graphed at the right. The open circle at 32 shows that all numbers to the left of 32 are graphed, but not 32 itself.

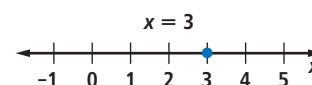**Exactly One Solution**

$x = 3$

Infinitely Many Solutions

$x > 3$

$x \geq 3$

32

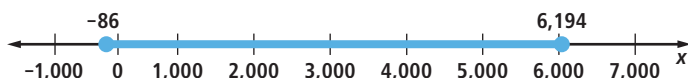**STOP** QY

Some situations lead to *double inequalities*. Let $E =$ the elevation of a place in the United States. Elevations in the United States range from 86 meters below sea level (in Death Valley) to 6,194 meters above sea level (at the top of Denali, also known as Mt. McKinley). So, $-86 \leq x \leq 6{,}194$. This double inequality is graphed below.

Draw dots on -86 and $6{,}194$ to show that those numbers are included in the solution.

The graph of $-86 \leq x \leq 6{,}194$ is an *interval*. An **interval** is a set of numbers between two numbers a and b, which are called the **endpoints** of the interval. When the endpoints are included in the interval, it is described by $a \leq x \leq b$. If the endpoints are not included, the $\leq$ is replaced by $<$, giving $a < x < b$.

The height of Mt. McKinley (Denali) is 20,320 feet.

Source: U.S. Geological Survey

Activity

The inequalities in 1–7 below are very similar. Use a CAS to solve each one. Record the operation you do to both sides and the inequality that results. A CAS screen for the first problem is shown.

1. $2x < 8$ 2. $2x < -8$ 3. $2x \leq 8$ 4. $-2x < 8$

5. $-2x \leq -8$ 6. $2x > 8$ 7. $-2x \geq 8$

In 8 and 9, give what you expect the solution to be. Use a CAS to check your answer.

8. $4m > -20$ 9. $-10w \leq 62$

10. The inequalities you solved in 1–9 can be grouped into two categories whose solution processes are somewhat different. What are the two categories?
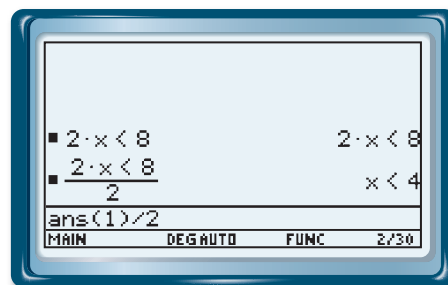
The Multiplication Property of Inequality

Here are some numbers in increasing order. Because the numbers are in order, you can put the inequality sign ($<$) between any two of them.
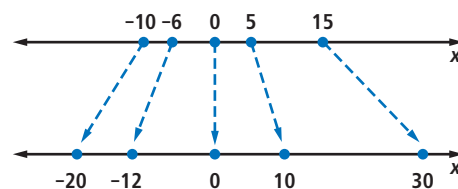
$$-10 < -6 < 0 < 5 < 15$$

Now multiply these numbers by some fixed *positive* number, say 2.

$$-20 \qquad -12 \qquad 0 \qquad 10 \qquad 30$$

The order stays the same, as shown in the number line at the right.

You could still put a $<$ sign between any two of the numbers. This illustrates that if $x < y,$ then $2x < 2y.$ In general, multiplication by a positive number maintains the order of a pair or a list of numbers.

> ### Multiplication Property of Inequality (Part 1)
>
> If $x < y$ and a is *positive*, then $ax < ay$.

Here is the same list of numbers we used earlier.

$$-10 < -6 < 0 < 5 < 15$$

Now we multiply these numbers by -2 and something different happens.

$$20 \qquad 12 \qquad 0 \qquad -10 \qquad -30$$

Notice that the numbers in the original list are in *increasing* order, while the numbers in the second list are in *decreasing* order. The order has been reversed.

If you multiply both sides of an inequality by a *negative* number, you must change the direction of the inequality. This idea can be generalized.
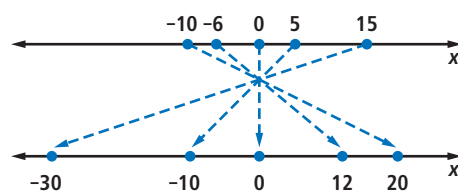
> ### Multiplication Property of Inequality (Part 2)
>
> If $x < y$ and a is *negative,* then $ax > ay$.

Changing from $<$ to $>$, or from $\leq$ to $\geq$, or vice-versa, is called *changing the sense of the inequality.* You have to change the sense of an inequality when you are multiplying both sides by a negative number. Otherwise, solving $ax < b$ is similar to solving $ax = b$.

Solving Inequalities

To solve an inequality of the form $ax < b,$ you must isolate the variable on one side (just like solving an equation). Be careful to notice whether you multiply or divide each side by a positive number or a negative number.

Example 1

Solve $-7x \geq 126$ and check.

Solution Multiply both sides by $-\frac{1}{7}$, the reciprocal of -7. Since $-\frac{1}{7}$ is a negative number, Part 2 of the Multiplication Property of Inequality tells you to change the sense of the inequality sign from $\geq$ to $\leq$.

$-7x \geq 126$	Write the inequality.
$-\frac{1}{7} \cdot -7x \leq -\frac{1}{7} \cdot 126$	Multiply each side by $-\frac{1}{7}$.
$x \leq -\frac{126}{7}$	Simplify.
$x \leq -18$	Simplify.

Check

Step 1 Substitute -18 for x. Does $-7 \cdot -18 = 126$? Yes.

Step 2 Try a number satisfying $x < -18$. We use -20. Is $-7 \cdot -20 \geq 126$? Yes, $140 \geq 126$.

> ▶ **READING MATH**
>
> Inequalities with variables are open sentences. When the variable is replaced with a number, the inequality is either true or false.

The two-step check of an inequality is important. The first step checks the boundary point in the solution. The second step checks the sense of the inequality.

GUIDED

Example 2

a. Solve $20 \geq 4x$.

b. Graph the solution.

c. Check your answer.

Solutions

a.
$20 \geq 4x$	Write the inequality.
$\frac{1}{4} \cdot 20 \underline{?} \frac{1}{4} \cdot 4x$	Multiply each side by $\frac{1}{4}$.
$5 \underline{?} x$	Simplify.

This can be rewritten as $x \underline{?} 5$.

b. Graph the solution on the number line.
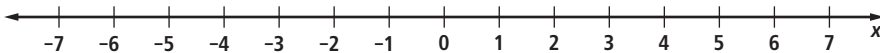
c. **Step 1** Check the boundary point by substituting 5 for x.
Does $\underline{?} = 4(\underline{?})$?

Step 2 Check whether the sense of the inequality is correct. Pick some number from the shaded region in Part b. This number should also work in the original inequality. We choose 0.

Is __?__ $\geq 4($__?__$)$? Yes, __?__ $\geq$ __?__.

Since both steps worked, the solution to $20 \geq 4x$ can be described by the sentence __?__.

Examples 1 and 2 showed solutions of two types of inequalities.

> ## Multiplication Property of Inequality (in words)
>
> You may multiply both sides of an inequality by the same *positive* number without affecting the set of solutions to the sentence. You can also multiply both sides by a *negative* number, but then you must also *change the sense of the inequality.*

Questions

COVERING THE IDEAS

In 1 and 2, consider the situation. a. Write an inequality that describes the situation. b. Identify the boundary point for the inequality. c. Graph the solution set of the inequality.

1. To successfully leave Earth's orbit, a satellite must be launched at a velocity of at least 11.2 kilometers per second. Let v be the launch velocity of a satellite in kilometers per second.

2. To obtain a passing grade on the Spanish exam, Marlene needs to answer at least 60 percent of the test items correctly. Let p be the percentage of items answered correctly.

3. Wakana's dog w weighs over 120 pounds, but not more than 130 pounds. Graph the possible values of w on a number line.

The SBS-4 communications satellite is deployed from the space shuttle payload bay.

Source: National Aeronautics and Space Administration

In 4 and 5, graph the inequality on a number line.

4. $x < 2\frac{1}{5}$

5. $y \leq -3$

6. Consider the inequality $20 < 30$. What true inequality results if you multiply both sides of the inequality by the given number?

   a. 6

   b. $\frac{1}{2}$

   c. -4

In 7–12, solve the inequality. Then check your answer.

7. $15x > 5$

8. $-32 < 2n$

9. $\frac{4}{5}y \leq 20$

10. $-\frac{3}{2}z \leq 1$

11. $-3a \leq -6$

12. $7b \geq 6$

13. Consider the following list of numbers. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$

   a. Place an appropriate symbol between each pair of consecutive numbers ($<$, $\leq$, $\geq$, or $>$).

   b. Multiply each number in the list by –8, then place an appropriate symbol between each pair of consecutive numbers ($<$, $\leq$, $\geq$, or $>$) in the new list.

   c. Did the direction of the symbols change in Part b? Why or why not?

## APPLYING THE MATHEMATICS

14. Consider the following list of numbers. 10, 20, 30, 40, 50

   a. Place an appropriate symbol between each pair of consecutive numbers ($<$, $\leq$, $\geq$, or $>$).

   b. Create a new list by taking the reciprocal of each number, then place an appropriate symbol between each pair of consecutive numbers ($<$, $\leq$, $\geq$, or $>$) in the new list.

   c. Did the direction of the symbols change in Part b? Why or why not?

15. **Fill in the Blanks** If $6 \leq m \leq 6.2$, then $\underline{\ ?\ } \leq 5m \leq \underline{\ ?\ }$.

In 16 and 17, read the problem situation. a. Write an inequality describing the situation. b. Solve the inequality.

16. An engineer is designing a rectangular parking lot that is to be 75 feet wide. According to the city building code, the area of the lot can be at most 6,250 square feet. What is the allowable length of the lot?

17. A concert hall is being designed to seat at least 2,200 people. Each row will have 80 seats. How many rows of seats will the hall have?

REVIEW

18. The boxes are of unknown equal weight W. (**Lesson 3-5**)
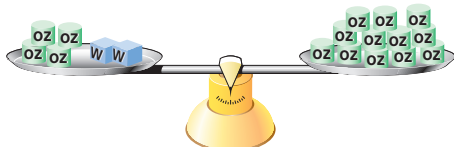
   a. What equation is pictured by the balance above?

   b. What two steps can be done with the weights on the balance to find the weight of a single box?

   c. How much does each box weigh?

In 19 and 20, solve the equation. (Lesson 3-5)

19. $-4(3x - 1.5) + 7 = 39$ **20.** $2(a + 5) - 3(5 + \frac{1}{2}a) = -19$

21. Grafton went to the store to buy bottles of soda and bags of chips for a party. He bought bottles of soda for $1.99 each and bags of chips for $2.99 each. He bought twice as many bags of chips as bottles of soda. After paying with two twenty-dollar bills, he received $0.15 in change. (Lessons 3-5, 3-3)
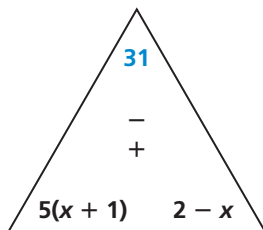
    a. Define a variable and write an equation describing the situation.

    b. How many bottles of soda and bags of chips did Grafton buy?

22. **Multiple Choice** How do the solutions to $2x - 111 = 35$ and $-35 = 2x + 111$ compare? (Lessons 3-4, 2-8, 2-4)
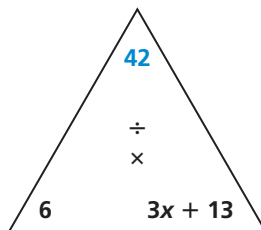
    A They are equal.   **B** They are opposites.

    C They are reciprocals.   **D** none of the above

In 23 and 24, a fact triangle is given. Write the related facts and determine the value of x. (Lesson 2-7)

23.

Triangle: 31 at top; − and + in middle; $5(x + 1)$ and $2 - x$ at bottom.

24.

Triangle: 42 at top; ÷ and × in middle; 6 and $3x + 13$ at bottom.

25. Tomás drove at an average speed of 61 miles per hour for $3\frac{1}{4}$ hours. About how many miles did he travel? (Previous Course)

EXPLORATION

26. A rectangle is 12 units by w units.

    a. Find the values of w that would make the area of the rectangle greater than 84 square units.

    b. Find the values of w that would make the area of the rectangle less than or equal to 216 square units.

    c. Write a sentence to explain what the inequality $60 \leq 12w < 108$ means in relation to the rectangle.

    d. Solve the inequality in Part c and explain its meaning.

w
12

QY ANSWER

$\{x: x < 32\}$