Lesson

10-9 Radian Measure

Vocabulary

radian

BIG IDEA The *radian* is an alternate unit of angle measure defined so the length of an arc in a unit circle is equal to the radian measure of that arc.

So far in this chapter you have learned to evaluate $\sin x$, $\cos x$, and $\tan x$ when x is given in degrees. Angles and magnitudes of rotations may also be measured in *radians*. In calculus and some other areas of mathematics, radians are used more often than degrees.

Mental Math

Ronin just returned from a trip and has several kinds of currency in his wallet. Assume there are 1.5 U.S. dollars in 1 euro, 100 Japanese yen in 1 U.S. dollar, and 0.75 South African rand in 10 Japanese yen.

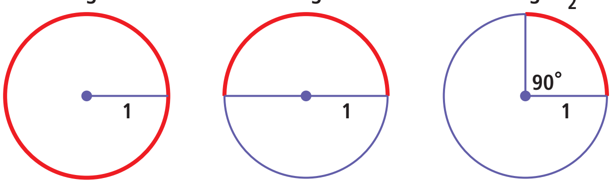
a. How many euros can Ronin get for his \$36?

b. About how many euros can he get for his 1700 yen?

c. How many U.S. dollars can he get for his 120 rand?

What Is a Radian?

Because the radius of a unit circle is 1, the circumference of a unit circle is 2π. Thus, on a unit circle, a $360°$ arc has length 2π. Similarly, a $180°$ arc has length $\frac{1}{2}(2\pi) = \pi$, and a $90°$ arc has length $\frac{1}{4}(2\pi) = \frac{\pi}{2}$.



The radian is a measure created so that the arc *measure* and the arc *length* are the same number.

Definition of Radian

The **radian** is a measure of an angle, arc, or rotation such that π radians $= 180$ degrees.

Notice that a $180°$ arc on the unit circle has measure π radians, and this arc has length π. A $90°$ angle has measure $\frac{\pi}{2}$ radians, and its arc has length $\frac{\pi}{2}$. In general, *the measure of an angle, arc, or rotation in radians equals the length of its arc on the unit circle.*

STOP QY

QY

What is the length of a $60°$ arc on the unit circle?

Conversion Factors for Degrees and Radians

The definition of radian can be used to create conversion factors for changing degrees into radians, and vice versa. Begin with the equation

$$\pi \text{ radians} = 180°.$$

Dividing each side by π radians gives $\frac{\pi \text{ radians}}{\pi \text{ radians}} = \frac{180°}{\pi \text{ radians}}$.

So, $1 = \frac{180°}{\pi \text{ radians}}$.

Similarly, dividing each side by 180° gives $\frac{\pi \text{ radians}}{180°} = \frac{180°}{180°}$,

so, $\frac{\pi \text{ radians}}{180°} = 1$.

Conversion Factors for Degrees and Radians

To convert radians to degrees, multiply by $\frac{180°}{\pi \text{ radians}}$.

To convert degrees to radians, multiply by $\frac{\pi \text{ radians}}{180°}$.

You may be wondering how big a radian is. Example 1 gives an answer.

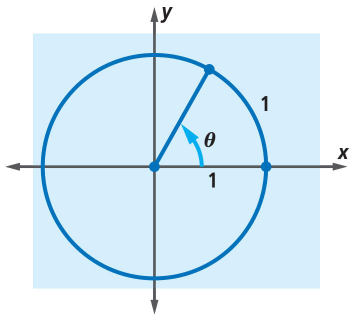
Example 1

Convert 1 radian to degrees.

Solution Because radians are given, multiply by the conversion factor with radians in the denominator.

$$1 \text{ radian} = 1 \text{ radian} \cdot \frac{180°}{\pi \text{ radians}}$$
$$= \frac{180°}{\pi}$$
$$\approx 57.3°$$

Notice that one radian is much larger than one degree.



Because the measure of an angle in radians equals the length of its arc on the unit circle, the angle of 1 radian in Example 1 determines an arc of length 1. A unit circle has circumference $2\pi \approx 6.28$, so an arc length of 1 is a little less than $\frac{1}{6}$ of the circle's circumference. So the angle $\theta = 1$ radian measures a little less than $\frac{1}{6}$ of 360°, or a little less than 60°.

GUIDED

Example 2

a. Convert 45° to its exact radian equivalent.

b. Convert $\frac{2}{3}\pi$ radians to its exact degree equivalent.

Solution

a. Multiply 45° by one of the conversion factors. Because you want radians, choose the ratio with radians in the numerator.

$$45° \cdot \underline{\quad ? \quad} = \underline{\ ?\ } \text{ radians}$$
$$= \underline{\ ?\ } \text{ radian}$$
$$45° = \underline{\ ?\ } \text{ radian, exactly.}$$

b. Multiply $\frac{2}{3}\pi$ radians by one of the conversion factors. Because you want degrees, choose the ratio with degrees in the numerator.

$$\frac{2}{3}\pi \cdot \underline{\quad ? \quad} = \underline{\quad ? \quad}$$
$$= \underline{\ ?\ }$$
$$\frac{2}{3}\pi = \underline{\ ?\ }\text{, exactly.}$$

Radian expressions are often left as multiples of π because this form gives an exact value. Usually in mathematics, the word *radian* or the abbreviation *rad* is omitted. In trigonometry, when no degree symbol or other unit is specified, we assume that the measure of the angle, arc, or rotation is radians.

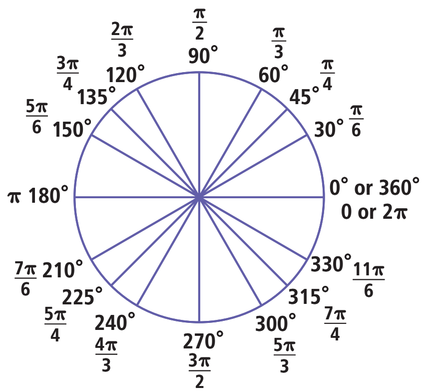
$\theta = 2°$ means
"the angle (or the arc or rotation) θ has measure 2 degrees."

$\theta = 2$ means
"the angle (or the arc or rotation) θ has measure 2 radians."

Refer to Guided Example 2. Because $\frac{\pi}{4} = 45°$, you can conclude that $\frac{3\pi}{4} = 3 \cdot \frac{\pi}{4} = 3 \cdot 45° = 135°$.

Similarly, $\frac{5\pi}{4} = 5 \cdot 45° = 225°$.

The diagram at right shows some common equivalences of degrees and radians.



In general, the multiples of π and the simplest fractional parts of π ($\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$, and their multiples) correspond to those angle measures that give exact values of sines, cosines, and tangents.

Trigonometric Values in Radians

Every scientific calculator can evaluate sin θ, cos θ, and tan θ, where θ is in radians.

GUIDED

Example 3

Set your calculator to radian mode. Evaluate

a. cos 2. b. tan $\frac{3\pi}{4}$.

Solution

a. Enter cos _?_. You should find cos _?_ $\approx$ _?_.

b. Enter tan _?_. You should find tan _?_ $=$ _?_.

Check

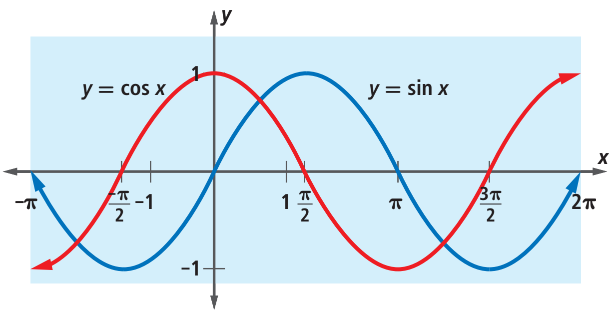
a. 1 radian $\approx$ _?_°, so 2 radians $\approx$ _?_°.

cos _?_° $\approx$ _?_. It checks.

b. $\frac{3\pi}{4}$ radians $=$ _?_°, and tan _?_° $=$ _?_. It checks.

Graphs of the Sine and Cosine Functions Using Radians

The cosine and sine functions are graphed below with x measured in radians rather than degrees. Notice that each function is (still) periodic, but that each period is 2π rather than 360°.



Questions

COVERING THE IDEAS

1. A circle has a radius of 1 unit. Give the length of an arc with the given measure.

   a. 360° b. 180° c. 90° d. 110°

2. **Fill in the Blanks**

   a. π radians $=$ _?_°

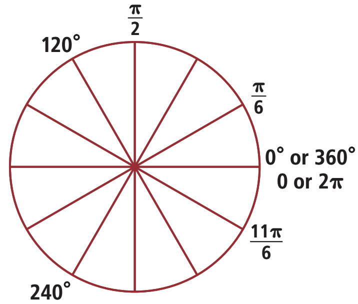
   b. 1 radian $=$ _?_°

   c. _?_ radian(s) $= 1°$

In 3–6, convert the radian measure to degrees.

3. 8π
4. $\frac{11\pi}{2}$
5. $\frac{14\pi}{45}$
6. $-\frac{5\pi}{4}$

In 7–10, convert to radians. Give your answer as a rational number times π.



7. $90°$ **8.** $15°$ **9.** $225°$ **10.** $330°$

11. a. Explain the different meanings of sin 6 and sin 6°.

b. Evaluate sin 6 and sin 6°.

12. Six equally-spaced diameters are drawn on a unit circle, as shown at the right. Copy and complete the diagram, giving equivalent measures in degrees and radians at the end of each radius.

In 13 and 14,

a. evaluate on a calculator in radian mode.

b. check your answer using degrees.

13. $\cos \frac{3\pi}{2}$ **14.** $\tan \frac{\pi}{6}$

In 15 and 16, suppose *x* is in radians.

15. What is the period of the function $y = \sin x$?

16. Name three x-intercepts of the function $y = \cos x$ on the interval $-2\pi < x < 2\pi$.

APPLYING THE MATHEMATICS

17. Multiple Choice Suppose x is in radians. Which transformations map the graph of $y = \cos x$ onto itself?

A reflection over the x-axis **B** reflection over the y-axis

C translation of π to the right **D** translation of 2π to the right

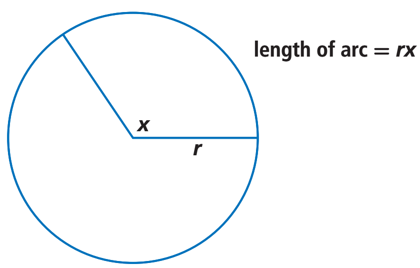
18. What is the measure of the obtuse angle made by the hands of a clock at 1:30

a. in degrees? **b.** in radians?

In 19 and 20, find the exact values.

19. $\sin\left(\frac{9\pi}{2}\right)$ **20.** $\cos\left(\frac{11\pi}{4}\right)$

In 21–23, use the following relationship between radian measure and arc length: In a circle of radius *r*, a central angle of *x* radians has an arc of length *rx*.



21. a. How long is the arc of a $\frac{\pi}{4}$-radian angle in a circle of radius 10?

b. How long is a 45° arc in a circle of radius 20?

22. On a circle of radius 2 meters, find the length of a 75° arc.

23. How long is the arc of a $\frac{2\pi}{3}$-radian angle in a circle of radius 8 feet?

REVIEW

24. In $\triangle ABC$, $m\angle B = 16°$, $b = 10$, and $c = 24$. Explain why there are two possible measures of $\angle C$, and find both of them. **(Lesson 10-8)**

25. Suppose $0° < \theta < 180°$. Solve $\sin \theta = 0.76$. **(Lesson 10-7)**

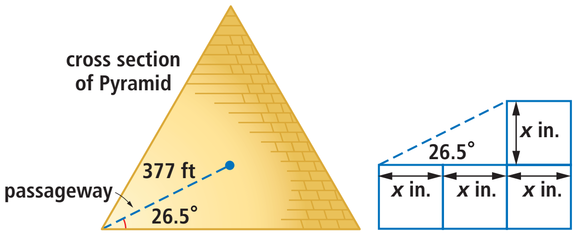
26. In $\triangle MAP$, $m = 22$, $m\angle M = 149°$, and $m\angle P = 23°$. Find the lengths of p and a. **(Lesson 10-7)**

In 27 and 28, evaluate the expression without using a calculator. (Lesson 10-5)

27. $(\sin 450°)^2 + (\cos 450°)^2$

28. $\tan 135°$

29. The newspaper article at the right is from the *Detroit Free Press*, January 29, 1985. Using the drawing below, explain why the construction technique leads to an angle of 26.5°. **(Lesson 10-2)**



30. One of Murphy's Laws says that the relative frequency r of rain is inversely proportional to the diameter d of the umbrella you are carrying. **(Lesson 2-2)**
    a. Write an equation relating r and d.
    b. If the probability of rain is $\frac{1}{4}$ when you are carrying a 46″-diameter umbrella, what is the probability when you are carrying a 36″-diameter umbrella?

EXPLORATION

31. When x is in radians, $\sin x$ can be estimated by the formula $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$.
    a. How close is the value of this expression to $\sin x$ when $x = \frac{\pi}{4}$?
    b. To get greater accuracy, you can add $\frac{x^9}{362{,}880}$ to the value you got in Part a. Where does the 362,880 come from?
    c. Add $\frac{\left(\frac{\pi}{4}\right)^9}{362{,}000}$ to your answer to Part a. How close is this value to $\sin \frac{\pi}{4}$?

Astronomer Solves Riddle of Pyramid

United Press International

A Navy astronomer has devised a surprisingly simple explanation for the angle of a descending passageway in the Great Pyramid of Cheops in Egypt.

In the early 19th century, English astronomer John Herschel suggested the 377-foot–long passageway was built at its angle of 26.5230 degrees to point at the North Star, making the pyramid an astronomical observatory as well as a tomb for Cheops.

Richard Walker, a U.S. Naval Observatory astronomer based at Flagstaff, Ariz., checked Herschel's idea and found that, because of the wobble of the Earth's axis in its orbit around the sun, no prominent star could have been seen from the base of the passageway built in 2800 BC when the pyramid was built.

Then why was the passageway inclined at an angle of 26.5 degrees?

According to Walker's report, the angle merely was the result of the construction technique.

By placing three stones of equal length horizontally and then placing a fourth stone of equal size on the top of the third horizontal stone, Walker determined that the angle from the top stone to the bottom stone at the other end is 26.5 degrees.

QY ANSWER

$\frac{\pi}{3}$