# Lesson 10-8 The Law of Cosines

**BIG IDEA** Given SAS or SSS in a triangle, the Law of Cosines enables you to find the lengths of the remaining sides or measures of angles of the triangle.

The Law of Sines enables you to solve a triangle if you know two angles and a side of the triangle (the ASA or AAS conditions). However, if you know the measures of two sides and the included angle (the SAS condition) or the measures of three sides of a triangle (the SSS condition), the Law of Sines cannot be used to solve the triangle. Fortunately, you can find other measures in the triangle using the *Law of Cosines*.

## Mental Math

**True or False**

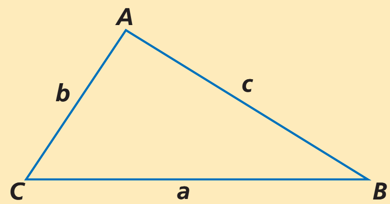
**a.** The length of your hair varies directly with the amount of time since your last haircut.

**b.** The length of your hair varies directly with the number of haircuts you've had this year.

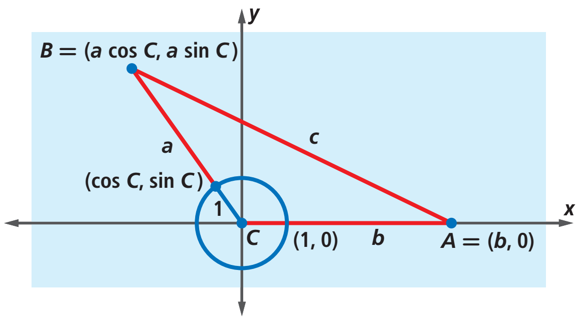
**c.** The amount of money you spend on haircuts each year varies jointly with the price of each haircut and the number of haircuts you have.

### Law of Cosines Theorem

In any triangle $ABC$,
$c^2 = a^2 + b^2 - 2ab \cdot \cos C$.




**Proof** Set up $\triangle ABC$ on a coordinate plane so that $C = (0, 0)$ and $A = (b, 0)$. To find the coordinates of point $B$, notice that $B$ is the image of $(\cos C, \sin C)$ under a size change of magnitude $a$. Thus, $B = (a \cos C, a \sin C)$. Recall the Pythagorean Distance Formula for the distance $d$ between $(x_1, y_1)$ and $(x_2, y_2)$, $d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$. Use it to find $c$ and square the result.




| | |
|---|---|
| $c = \sqrt{\lvert a \cos C - b\rvert^2 + \lvert a \sin C - 0\rvert^2}$ | Pythagorean Distance Formula |
| $c^2 = \lvert a \cos C - b\rvert^2 + \lvert a \sin C - 0\rvert^2$ | Square both sides. |
| $c^2 = a^2\cos^2 C - 2ab \cdot \cos C + b^2 + a^2\sin^2 C$ | Expand the binomials. |
| $c^2 = a^2\cos^2 C + a^2\sin^2 C + b^2 - 2ab \cdot \cos C$ | Commutative Property of Addition |
| $c^2 = a^2(\cos^2 C + \sin^2 C) + b^2 - 2ab \cdot \cos C$ | Distributive Property |
| $c^2 = a^2 + b^2 - 2ab \cdot \cos C$ | Pythagorean Identity |

The Law of Cosines applies to any two sides of a triangle and their included angle. So it is also true that in $\triangle ABC$,

$$a^2 = b^2 + c^2 - 2bc \cos A \text{ and } b^2 = a^2 + c^2 - 2ac \cos B.$$

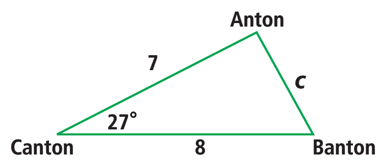
In words the Law of Cosines says that in any triangle, the sum of the squares of two sides minus twice the product of these sides and the cosine of the included angle equals the square of the third side.

## Using the Law of Cosines to Find a Length

With the Law of Cosines, finding the length of the third side of a triangle when two sides and the included angle are known requires only substitution.

### Example 1

**Two straight roads meet in Canton at a 27° angle. Anton is 7 miles down one road, and Banton is 8 miles down the other. How far apart are Anton and Banton?**




**Solution 1** Because this is an SAS situation, use the Law of Cosines. Let $c$ be the distance between Anton and Banton.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Substitute. $c^2 = 8^2 + 7^2 - 2 \cdot 8 \cdot 7 \cdot \cos 27°$

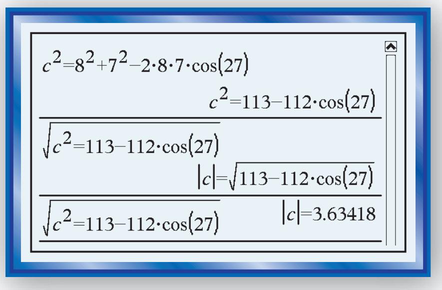
$$c^2 \approx 13.207$$

$$c \approx \pm 3.63$$

Because lengths of sides of triangles must be positive numbers, only the positive solution makes sense in this situation. So $c \approx 3.63$. Anton and Banton are about 3.6 miles apart.

**Solution 2** Enter the equation into a CAS and then take the square root of the result.

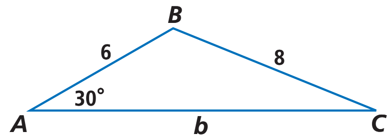
This display shows both exact and approximate solutions. They agree with Solution 1.




You can also use the Law of Cosines to find a length in triangles that meet the SsA condition, where the lengths of two sides are known, along with the measure of the angle opposite the longer side.

## Example 2

**In $\triangle ABC$, $c = 6$ cm, $a = 8$ cm, and $m\angle A = 30°$. Find $b$.**

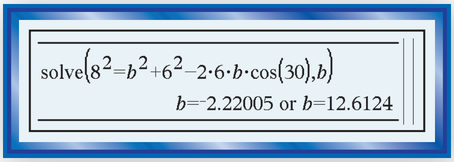



**Solution** This SsA situation indicates that you can use the Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$8^2 = b^2 + 6^2 - 2 \cdot 6 \cdot b \cdot \cos 30°$$

Solve on a CAS.




Because side lengths must be positive numbers, $b \approx 12.6$ cm.

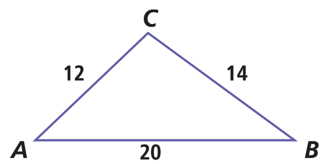
## Using the Law of Cosines to Find an Angle

You can also use the Law of Cosines to find the measure of any angle in a triangle when you know the lengths of all three sides (the SSS condition).

**GUIDED**

## Example 3

**A city wants to build a grass-covered playground on a small triangle-shaped lot with boundaries of length 12 m, 14 m, and 20 m. Through what angle measure should an automatic sprinkler be set to water the grass if it is placed at the corner $C$?**




**Solution** In this SSS situation, you want to find $m\angle C$. Use the Law of Cosines.

Let $a = \underline{\ ?\ }$, $b = \underline{\ ?\ }$, and $c = \underline{\ ?\ }$.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Substitute. $\underline{\ ?\ }^2 = \underline{\ ?\ }^2 + \underline{\ ?\ }^2 - 2 \cdot \underline{\ ?\ } \cdot \underline{\ ?\ } \cos C$

$\underline{\ ?\ } = \underline{\ ?\ } - \underline{\ ?\ } \cos C$

Solve for $\cos C$. $\underline{\ ?\ } = \underline{\ ?\ } \cos C$

$\cos C \approx \underline{\ ?\ }$

Apply $\cos^{-1}$ to both sides. $m\angle C \approx \underline{\ ?\ }$

The sprinkler should be set to cover an angle of about $\underline{\ ?\ }$.

STOP QY

**QY**

Check your solution to Example 3 by substituting $a$, $b$, $c$, and your solution for $m\angle C$ into $a^2 + b^2 - 2ab \cos C = c^2$.

The Law of Cosines ($a^2 + b^2 - 2ab\cos C = c^2$) is the Pythagorean Theorem ($a^2 + b^2 = c^2$) with an extra term, $-2ab \cos C$. Consider three different triangles:

- If $\angle C$ is acute, as in Example 1, then $\cos C$ is positive and the extra term, $-2ab \cos C$, is negative. So $c^2 < a^2 + b^2$.
- If $\angle C$ is obtuse, as in Example 3, then $\cos C$ is negative and the extra term, $-2ab \cos C$, is positive. So $c^2 > a^2 + b^2$.
- If $\angle C$ is a right angle, then the extra term, $-2ab \cos 90°$, is equal to 0, and $c^2 = a^2 + b^2$.

This shows that the Law of Cosines is a generalization of the Pythagorean Theorem.

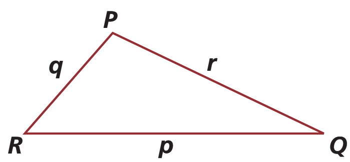
The last two lessons can be summarized as follows. If you need to find a side or an angle of a triangle, use the simplest methods possible before using trigonometry. If you still have sides or angles to find:

- Use right-angle trigonometric ratios when the missing side or angle is part of a right triangle.
- Use the Law of Sines for triangles meeting the ASA or AAS conditions.
- Use the Law of Cosines for triangles meeting the SAS, SSS, or SsA conditions.

## Questions

### COVERING THE IDEAS

**In 1–3, according to the Law of Cosines, what expression is equal to the following in $\triangle PQR$?**




1. $p^2 + r^2 - 2pr \cos Q$
2. $r^2 + q^2 - 2rq \cos P$
3. $r^2$

4. **Multiple Choice** Which of the following describes the Law of Cosines?

   A The third side of a triangle equals the sum of the squares of the other two sides minus the product of the two sides and the included angle.

   B The square of the third side of a triangle equals the sum of the squares of the other two sides minus the product of the two sides and the cosine of the included angle.

   C The square of the third side of a triangle equals the sum of the squares of the other two sides minus twice the product of the two sides and the cosine of the included angle.

   D none of the above

5. In $\triangle ABC$, $m\angle A = 27.5°$, $AB = 10$, and $AC = x$. Write an expression for the length of $BC$.

6. Refer to Example 1. Danton is a town on the road between Canton to Banton, 5 miles from Canton, with a straight road connecting Danton and Anton. What is the direct distance from Danton to Anton?

7. Check the answer to Example 2 by following these steps.

**Step 1** Solve $\frac{\sin A}{a} = \frac{\sin C}{c}$ for $m\angle C$.

**Step 2** Find $m\angle B$.

**Step 3** Solve $\frac{\sin A}{a} = \frac{\sin B}{b}$ for $b$.

8. Refer to Guided Example 3. Find $m\angle B$.

9. For $\triangle ABC$, use the Law of Cosines to prove that, if $\angle C$ is acute, then $a^2 + b^2 > c^2$.

## APPLYING THE MATHEMATICS

10. In $\triangle ABC$, $m\angle A = 53°$, $m\angle B = 102°$, and $m\angle C = 25°$. Explain why this triangle cannot be solved.

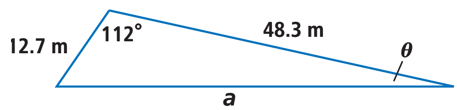
11. In $\triangle ABC$, use the Law of Cosines to get a formula for $\cos A$ in terms of $a$, $b$, and $c$.

12. An airplane flies the 261 miles from Albany, New York, to Buffalo, New York. It then changes its direction by turning 26° left and flies a distance of 454 miles to Chicago, Illinois. What is the direct distance from Albany to Chicago?

13. Refer to the triangle at the right.

a. Find $a$.

b. Find $\theta$.




14. The diagonals of a rectangle are 40 centimeters long and intersect at an angle of 28.5°. How long are the sides of the rectangle?

15. The sides of a rectangle are 4 inches and 7 inches. Find the measure of the acute angle formed by the diagonals.

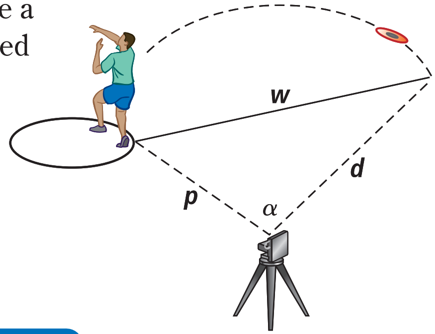
16. At a criminal trial, a witness gave the following testimony: "The defendant was 25 feet from the victim. I was 65 feet from the defendant and about 100 feet from the victim when the robbery occurred. I saw the whole thing."

a. Draw a triangle to represent the situation.

b. Use the Law of Cosines to show that the testimony has errors.

c. How else could you know that the testimony has errors?

17. At a track meet, an electronic device measures the distance a discus travels to the nearest centimeter. The device is placed as shown in the diagram at the right. It first measures the distance $p$ to the discus circle. After the athlete throws the discus, the device measures angle $\alpha$ and distance $d$ and calculates the length $w$ of the throw to the nearest centimeter. Suppose $p = 3.2$ m, $\alpha = 147.207°$ and $d = 47.40$ meters. How long was the throw?

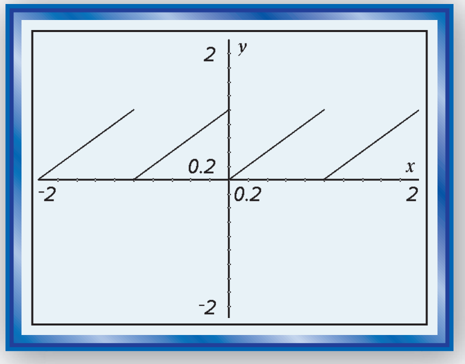



## REVIEW

18. In $\triangle DOG$, $OG = 42$, $m\angle D = 118°$, and $m\angle G = 27°$. Find $DO$ and $DG$. **(Lesson 10-7)**

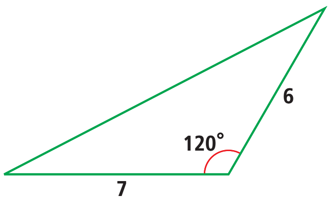
19. Does the graph of the function $f(x) = \sin x$ have any lines of symmetry? If so, give an equation for one such line. **(Lesson 10-6)**

20. a. Is the relationship graphed at the right a function? Why or why not? **(Lessons 10-6, 8-2, 1-2)**
    b. Is the relation periodic? If so, what is the period?
    c. Is the inverse of the relation a function? Why or why not?




21. Use the Pythagorean Identity to prove that the point with coordinates $(r \cos \theta, r \sin \theta)$ has distance $|r|$ from the origin. **(Lesson 10-5, Previous Course)**

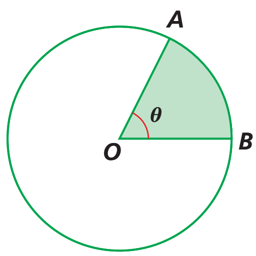
22. Find the area of the triangle at the right. **(Lesson 10-1)**




**In 23 and 24, refer to circle *O* at the right. (Previous Course)**

23. If $\theta = 85°$, what fraction of the circle's area is the area of the shaded sector?

24. If the length of $\overset{\frown}{AB}$ is $\frac{9}{24}$ of the circumference of the circle, find $\theta$.




## EXPLORATION

25. "You, who wish to study great and wonderful things, who wonder about the movement of the stars, must read these theorems about triangles. Knowing these ideas will open the door to all of astronomy and to certain geometric problems." This quotation is from *De Triangulis Omnimodis* by Regiomontanus. Find out more about this 15th-century mathematician and his work in trigonometry.

**QY ANSWERS**

$14^2 + 12^2 - 2 \cdot 14 \cdot 12 \cos 100° \approx 398 \approx 400 = 20^2$. It checks.