Chapter 10

Lesson

10-7 The Law of Sines

Vocabulary

solving a triangle

BIG IDEA Given AAS or ASA in a triangle, the Law of Sines enables you to find the lengths of the remaining sides.

One of the most important applications of trigonometry is to find unknown or inaccessible lengths or distances. In previous lessons, you learned to use trigonometric ratios to find unknown sides or angles of *right* triangles. In this lesson and the next, you will see how to find unknown sides or angles in *any* triangle, given enough information to determine the triangle. Using trigonometry to find all the missing measures of sides and angles of a triangle is called **solving the triangle**.

Mental Math

How many lines of symmetry does the graph of each equation have?

a. $y = -x^2$

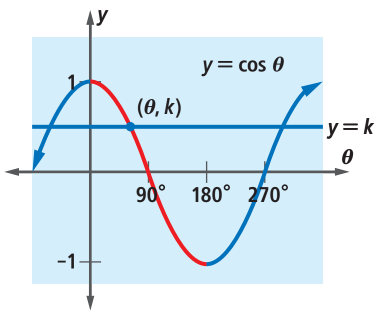
b. $y = -x^3$

c. $y = -x$

Solutions to cos $\theta = k$ When $0° < \theta < 180°$

Although the sine and cosine functions are periodic over the domain of real numbers, it is important to remember that if θ is an angle in a triangle, then $0° < \theta < 180°$.

When $0° < \theta < 180°$, the equation $\cos\theta = k$ has a unique solution. To see why, consider the graph of $y = \cos\theta$ on this interval. For any value of k between -1 and 1, the graph of $y = k$ intersects $y = \cos\theta$ at a single point. The θ-coordinate of this point is the solution to $\cos\theta = k$.



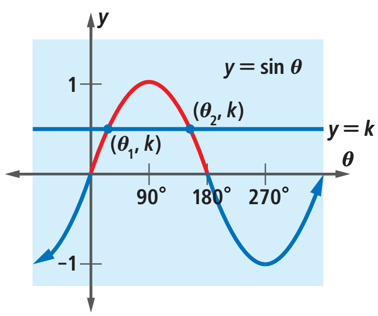
STOP QY1

QY1

Solve $\cos\theta = -\frac{1}{2}$ when $0° < \theta < 180°$ using the inverse cosine function.

Solutions to sin $\theta = k$ When $0° < \theta < 180°$

The situation is different for the equation $\sin\theta = k$. On the interval $0° < \theta < 180°$, for any value of k between 0 and 1, the graph of $y = k$ intersects $y = \sin\theta$ in two points. In the graph at the right, we call these points (θ_1, k) and (θ_2, k). The numbers θ_1 and θ_2 are the solutions to $\sin\theta = k$.



The points (θ_1, k) and (θ_2, k) are reflection images of each other over the vertical line with equation $\theta = 90°$. This is because when $0 < k < 1$, the two solutions to $\sin \theta = k$ between 0° and 180° are supplementary angles.

Supplements Theorem

For all θ in degrees, $\sin \theta = \sin(180° - \theta)$.

The Supplements Theorem allows you to solve equations of the form $\sin \theta = k$ without graphing.

Example 1

Find all solutions to $\sin \theta = 0.842$ in the interval $0° < \theta < 180°$.

Solution Use the inverse sine function to find a solution between 0° and 90°. $\theta_1 = \sin^{-1} 0.842 \approx 57.4°$

The second solution is the supplement of the first.

$\theta_2 = 180° - \theta_1 \approx 180° - 57.4° = 122.6°$

So, when $\sin \theta = 0.842$, $\theta \approx 57.4°$ or $122.6°$.

Check $\sin 57.4° \approx 0.842$ and $\sin 122.6° \approx 0.842$. They check.

STOP **QY2**

The Law of Sines

Activity

MATERIALS ruler, protractor

Work with a partner.

Step 1 Each partner should draw a different triangle *ABC* on a sheet of notebook paper. Measure the side lengths to the nearest tenth of a centimeter and the angles as accurately as you can.

Step 2 Let *a*, *b*, and *c* be the lengths of the sides opposite angles *A*, *B*, and *C*, respectively. Use a calculator to compute $\frac{\sin A}{a}$, $\frac{\sin B}{b}$, and $\frac{\sin C}{c}$ to the nearest hundredth.

Step 3 Compare your results to those of your partner. What do you notice?

The results of the Activity are instances of a theorem that enables a triangle to be solved when the measures of two angles and one side are known.

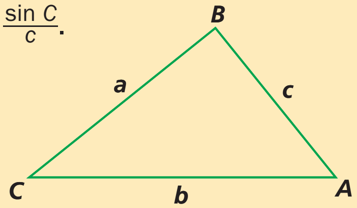
QY2

a. Solve $\sin \theta = -\frac{1}{2}$ when $0° < \theta < 180°$ using the inverse sine function or the graph of $y = \sin x$.

b. Solve $\sin \theta = \frac{1}{2}$ when $0° < \theta < 180°$ using the inverse sine function or the graph of $y = \sin x$.

Law of Sines Theorem

In any triangle ABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

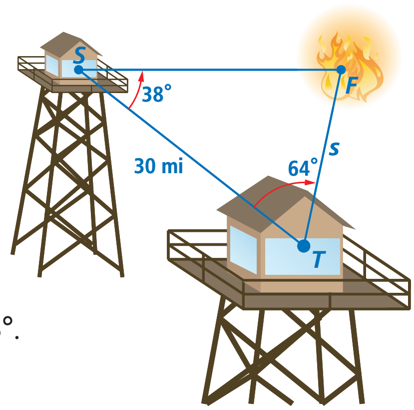


The Law of Sines states that in any triangle, the ratios of the sines of its angles to the lengths of the sides opposite them are equal. A proof of the theorem is given below. You are asked in Question 13 to fill in the missing information.

Proof Recall from Question 15 of Lesson 10-1 that the area of a triangle is $\frac{1}{2}$ the product of two sides and the sine of the included angle. Consequently, area($\triangle ABC$) $= \frac{1}{2}ab \sin C$, area($\triangle ABC$) $= \frac{1}{2}ac \sin B$, and area($\triangle ABC$) $= \frac{1}{2}bc \sin A$. So, by the ___?___, $\frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$. Multiply all three parts of this equation by 2. The result is ___?___. Divide all three parts by abc. You get ___?___. Simplify the fractions to get $\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$.

Example 2

Two forest rangers are in their stations, S and T, 30 miles apart. On a certain day, the ranger at S sees a fire at F, at an angle of 38° with segment $\overline{ST}$. The ranger at T sees the same fire at an angle of 64° with $\overline{ST}$. Find the distance from station T to the fire.



Solution Let s be the desired distance. The angle opposite s is $\angle S$, with measure 38°. To use the Law of Sines, you need the measures of another angle and its opposite side. Because the sum of the measures of the angles in a triangle is 180°, $\angle F$ has measure 78°. Now there is enough information to use the Law of Sines.

$$\frac{\sin S}{s} = \frac{\sin F}{f}$$

$$\frac{\sin 38°}{s} = \frac{\sin 78°}{30} \qquad \text{Substitution}$$

$$s = \frac{30 \sin 38°}{\sin 78°} \qquad \text{Solve for } s.$$

$$s \approx \frac{30(0.616)}{0.978} \approx 18.9$$

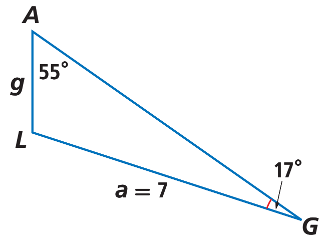
The fire is about 19 miles from station T.

The Law of Sines also can be used to find lengths in triangles when you know two angles and an adjacent side, the AAS (Angle-Angle-Side) condition from geometry.

GUIDED

Example 3

In $\triangle ALG$ at the right, $m\angle A = 55°$, $m\angle G = 17°$, and $a = 7$. Find g.



Solution

$\frac{\sin A}{a} = \frac{\sin G}{g}$ Law of Sines

$\underline{\ ?\ } = \underline{\ ?\ }$ Substitution

$g \approx \underline{\ ?\ }$ Solve for g.

Ptolemy knew of the Law of Sines in the 2nd century CE. This led eventually to *triangulation*, the process of dividing a region into triangular pieces, making a few accurate measurements, and using trigonometry to determine most of the unknown distances. This made it possible for reasonably accurate maps of parts of Earth to be drawn well before the days of artificial satellites. Now, Global Positioning Systems (GPS) use triangulation along with accurate data about the contours of Earth to produce highly accurate maps and to calculate the precise latitude and longitude of GPS devices.

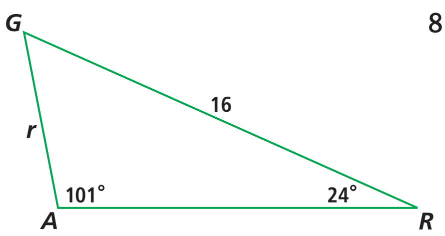
Questions

COVERING THE IDEAS

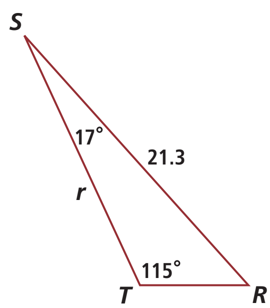
1. Explain why the equation $\sin \theta = k$ for $0° < \theta < 180°$ and $0 < k < 1$ has two solutions.
2. Solve $\sin \theta = 0.954$ for $0° < \theta < 180°$ by graphing.
3. Since $\sin 45° = \frac{\sqrt{2}}{2}$, for what other value of θ with $0° < \theta < 180°$ does $\sin \theta = \frac{\sqrt{2}}{2}$?
4. Explain how the Activity illustrates the Law of Sines.
5. Write a description of the Law of Sines as if you were explaining it to a friend.
6. Refer to Example 2. Find the distance from the fire to station S.

In 7 and 8, find r.

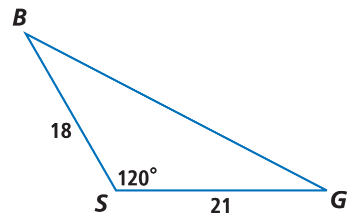
7.



8.



APPLYING THE MATHEMATICS

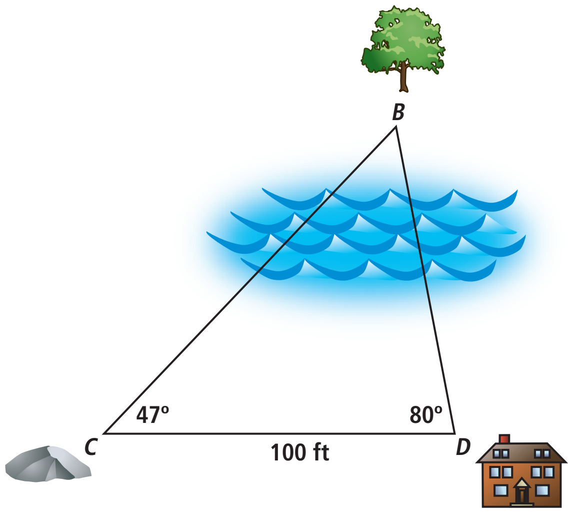


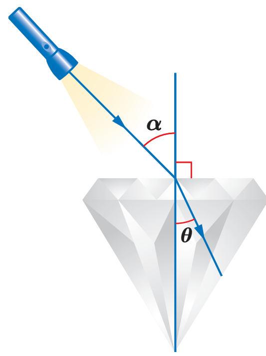
9. Refer to $\triangle BSG$ at the right.

a. Why is the statement $\sin 120° = \frac{21}{BG}$ incorrect?

b. Can you use the Law of Sines to find BG? Explain your answer.

10. Refer to the picture below. A rock at C and a house at D are 100 feet apart. A tree is across the river at B. $m\angle BCD = 47°$ and $m\angle BDC = 80°$. Find the distance across the river from the house to the tree.





11. When a beam of light traveling through air strikes the surface of a diamond, it is *refracted*, or bent, as shown at the right. The relationship between α and θ is known as Snell's Law,

$$\frac{\sin \alpha}{\text{speed of light in air}} = \frac{\sin \theta}{\text{speed of light in diamond}}.$$

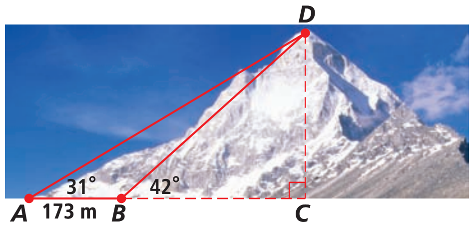
The speed of light in air is about $3 \cdot 10^8$ meters per second.

a. If $\alpha = 45°$ and $\theta = 17°$, estimate the speed of light in diamond.

b. Diamonds are identified as authentic by their *refractive index*. The refractive index of an object is the ratio of the speed of light in air to the speed of light in that object. Find the refractive index for a diamond.

c. Cubic zirconia, a lab-created alternative to diamond, has a refractive index of 2.2. Estimate the speed of light through cubic zirconia.

12. Because surveyors cannot get inside a mountain, every mountain's height must be measured indirectly. Refer to the diagram at the right. Assume all labeled points lie in a single plane.
    a. Find the measures of $\angle ABD$ and $\angle ADB$ without trigonometry.
    b. Find BD.
    c. Find DC, the height of the mountain.



13. Fill in the blanks in the proof of the Law of Sines on page 702.

REVIEW

14. **Fill in the Blank** A function is periodic if its graph can be mapped onto itself under a $\underline{\quad ? \quad}$. (Lesson 10-6)

15. Name three x-intercepts of the graph of $y = \cos x$. (Lesson 10-6)

Fill in the Blank In 16 and 17, complete each statement with a trigonometric expression to make the equation true for all θ. (Lesson 10-5)

16. $\cos^2 \theta + \underline{\quad ? \quad} = 1$

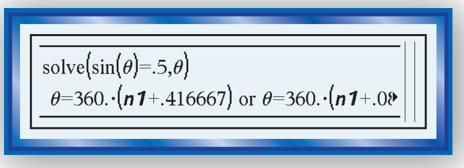
17. $\cos \theta = \underline{\quad ? \quad}$

18. Estimate to the nearest thousandth. (Lessons 9-8, 9-7, 9-5)
    a. $\log 49$
    b. $\ln 49$
    c. $\log_2 49$

EXPLORATION

19. Use a DGS to draw a circle with diameter $\overline{AB}$ and mark points $C, D, E,$ and F not equally spaced on the circle.
    a. Draw $\triangle ABC$ and measure to find $\frac{\sin A}{a}$, $\frac{\sin B}{b}$, and $\frac{\sin C}{c}$.
    b. Draw $\triangle DEF$ and measure to find $\frac{\sin D}{d}$, $\frac{\sin E}{e}$, and $\frac{\sin F}{f}$.
    c. What do you notice about your results in Parts a and b? What is the connection between the ratios and the length of the diameter?

20. In QY 2, you were asked to solve $\sin \theta = 0.5$ using its inverse function or the graph of $y = \sin x$.
    a. Solve $\sin \theta = 0.5$ on a CAS.
    b. One CAS gives the results at the right.
       `θ = 360 · (n1 + .416667)` or
       `θ = 360 · (n1 + .083333)`
       Interpret what the CAS display means.



QY ANSWERS

1. $\theta = 120°$

2. a. There are no solutions.
   b. $\theta = 30°$ or $150°$