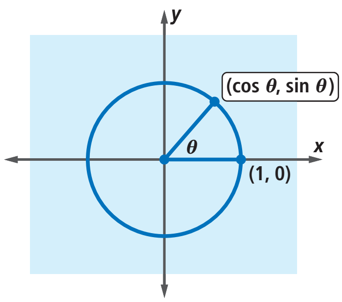
Lesson 10-6 The Cosine and Sine Functions

Vocabulary

periodic function, period
sine wave
sinusoidal

▶ **BIG IDEA** The graphs of the cosine and sine functions are sine waves with period 2π.

Remember that when (1, 0) is rotated θ degrees around the origin, its image is the point $(\cos \theta, \sin \theta)$. The correspondence $\theta \rightarrow \cos \theta$ is the cosine function, with domain the set of real numbers. The values of this function are the first coordinates of the images of (1, 0) under rotations about the origin.



Similarly, the correspondence $\theta \rightarrow \sin \theta$ is the sine function. The values of this function are the second coordinates of the images of (1, 0) under R_θ.

Mental Math

Which is longer?

a. the side of a regular octagon or its shortest diagonal

b. the leg opposite a 40° angle in a right triangle or the other leg

c. the diagonal of a square or the diameter of a circle inscribed in it

d. the diagonal of a square or the diameter of a circle circumscribed around it

A Graph of the Cosine Function

To imagine the graph of $y = \cos \theta$ as θ increases from 0, think of a point moving around the unit circle counterclockwise from (1, 0). As the point moves halfway around the circle, its first coordinate decreases from 1 to –1. As the point continues to move around the circle, its first coordinate increases from –1 to 1. The Activity provides more detail.

Activity

MATERIALS calculator

Set a calculator to degree mode.

Step 1 Make a table of values of $\cos \theta$ for values of θ in the interval $0° \le \theta \le 360°$, in increments of 15°. You will need 25 rows, not 10 as shown at the right. Round the cosines to the nearest hundredth. The first few pairs in the table are shown.

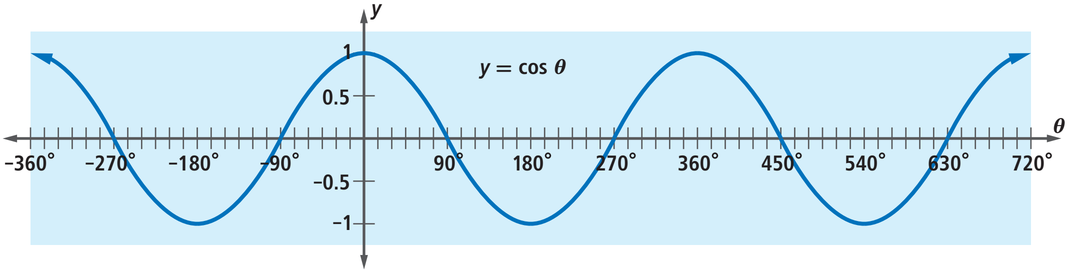
Step 2 Graph the points you found in Step 1. Plot θ on the horizontal axis and $\cos \theta$ on the vertical axis. Connect the points with a smooth curve.

Step 3 Describe two patterns you notice in your graph.

Step 4 What is the largest value that $\cos \theta$ can have? What is the smallest value that $\cos \theta$ can have?

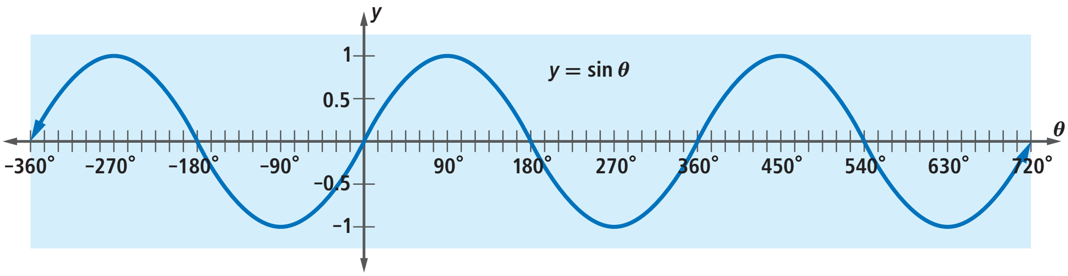
θ	$\cos \theta$
0°	1.00
15°	0.97
30°	0.87
45°	0.71
60°	?
75°	?
90°	?
⋮	⋮
345°	?
360°	?

Recall that as θ takes on values greater than 360°, cos θ repeats its values. So the graph of $y = \cos\theta$ repeats every 360°. Below is a graph of this function when $-360° \leq \theta \leq 720°$.



The Graph of the Sine Function

The graph of the sine function is constructed by a similar process, using the second coordinate of the rotation image of (1, 0) as the dependent variable. For instance, $R_{60}(1, 0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so $\sin 60° = \frac{\sqrt{3}}{2}$ and the point $\left(60°, \frac{\sqrt{3}}{2}\right)$ is on the graph of the sine function. Below is a graph of $y = \sin\theta$. Notice that the graph of the sine function looks congruent to the graph of the cosine function.



Properties of the Sine and Cosine Functions

A function is a **periodic function** if its graph can be mapped onto itself by a horizontal translation. Algebraically, this means that a function f is periodic if there is a positive number p such that $f(x + p) = f(x)$ for all values of x. The smallest positive number p with this property is called the **period** of the function. Both the sine and cosine functions are periodic because their values repeat every 360°. That is, for all θ, $\sin(\theta + 360°) = \sin\theta$ and $\cos(\theta + 360°) = \cos\theta$. This means that under a horizontal translation of magnitude 360°, the graph of $y = \sin\theta$ coincides with itself. Similarly, under this translation, the graph of $y = \cos\theta$ coincides with itself.

Notice that each of these functions has range $\{y \mid -1 \leq y \leq 1\}$. Also, each function has infinitely many x-intercepts, but still only one y-intercept. These and other properties of sine and cosine functions are summarized in the table on the next page.

	Cosine Function	Sine Function
Domain	set of all real numbers	set of all real numbers
Range	$\{y \mid -1 \le y \le 1\}$	$\{y \mid -1 \le y \le 1\}$
***x*-intercepts**	odd multiples of 90° $\{\ldots, -90°, 90°, 270°, 450°, \ldots\}$	even multiples of 90° $\{\ldots, -180°, 0°, 180°, 360°, \ldots\}$
Period	360°	360°
***y*-intercept**	1	0

The graph of the cosine function can be mapped onto the graph of the sine function by a horizontal translation of 90°. So the graphs of $y = \cos\theta$ and $y = \sin\theta$ are congruent. Both graphs are called *sine waves*.

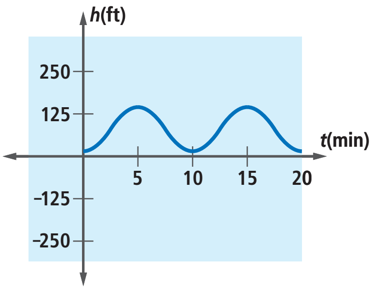
Definition of Sine Wave

A **sine wave** is a graph that can be mapped onto the graph of the sine function $s: \theta \to \sin\theta$ by any composite of translations, scale changes, or reflections.

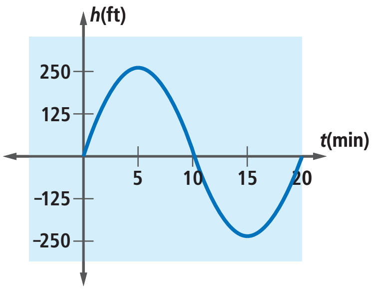
Because the graph of the cosine function $c: \theta \to \cos\theta$ is a translation image of the graph of $s: \theta \to \sin\theta$, its graph is a sine wave. Situations that lead to sine waves are said to be **sinusoidal**.

Example

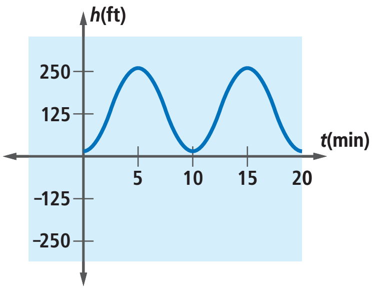
The Great Ferris Wheel built in 1893 for the Columbian Exposition in Chicago had a 125-foot radius and a center that stood 140 feet off the ground. A ride on the wheel took about 20 minutes and allowed the rider to reach the top of the wheel twice. Assume that a ride began at the bottom of the wheel and did not stop. Which sine wave below models the rider's height *h* off the ground *t* minutes after the ride began? Explain your choice.



Graph A



Graph B



Graph C

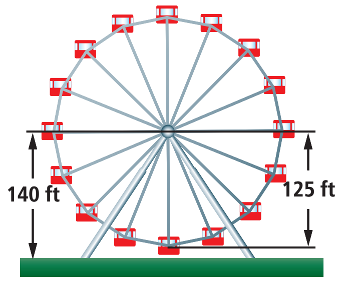
(continued on next page)

Solution Find the minimum and maximum height of a ride on the Ferris wheel.

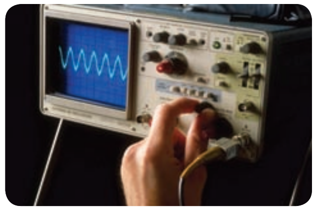
The minimum height is the difference between the height of the center of the wheel and its radius, or 140 ft – 125 ft = 15 ft. The maximum height is then 250 ft + 15 ft = 265 ft, the diameter of the wheel plus the wheel's height off the ground.

Find when the minimum and maximum height of a ride occurred.

The maximum height occurred twice in 20 minutes. A complete revolution took 10 minutes, so, the rider is 15 feet high at $t = 0$, 10, and 20 minutes. Without stops, the rider reached the top at $t = 5$ and 15 minutes. So, graph C is the correct graph.



Sine waves occur frequently in nature: in ocean waves, sound waves, and light waves. Also, the graph of average daily temperatures for a specific location over the year often approximates a sine wave. The voltages associated with alternating current (AC), the type used in electrical transmission lines, have sinusoidal graphs. Sine waves can be converted to electrical signals and then viewed on an oscilloscope.



An oscilloscope can be used to test electronic equipment.

Questions

COVERING THE IDEAS

1. What function maps θ onto the first coordinate of the image of (1, 0) under R_θ?

2. What function maps θ onto the second coordinate of the image of (1, 0) under R_θ?

In 3–5, consider the cosine function.

3. a. **Fill in the Blanks** As θ increases from 0° to 90°, cos θ decreases from __?__ to __?__.

   b. **Fill in the Blanks** As θ increases from 90° to 180°, cos θ decreases from __?__ to __?__.

   c. As θ increases from 180° to 270°, does the value of cos θ increase or decrease?

4. Name two points on the graph of the function when $\theta > 360°$.

5. How many solutions are there to the equation $\cos\theta = 0.5$ if $-720° \le \theta \le 720°$?

In 6–8, consider the sine function.

6. Explain why its period is 360°.

7. Name all θ-intercepts between -360° and 360°.

8. How many solutions are there to the equation $\sin\theta = 2$ if $-720° \le \theta \le 720°$?

In 9–11, true or false.

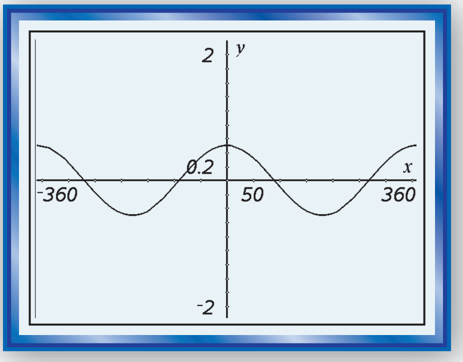
9. The graph of the cosine function is called a cosine wave.
10. The graph of the sine function is the image of the graph of the cosine function under a horizontal translation of 180°.
11. The ranges of the sine and cosine function are identical.
12. Refer to the Example.
    a. Describe the Ferris Wheel ride shown by graph A.
    b. Why does graph B not describe a Ferris Wheel ride?

APPLYING THE MATHEMATICS

13. Use the graphs of the sine and cosine functions.
    a. Find two values of θ, one positive and one negative, such that $\cos \theta > \sin \theta$.
    b. Name two values of θ for which $\cos \theta = \sin \theta$.
14. Consider these situations leading to periodic functions. What is the period?
    a. days of the week
    b. the ones digit in the successive integers in base 10

In 15–18, part of a function is graphed. Does the function appear to be periodic? If so, what is the period? If not, why not?

15.



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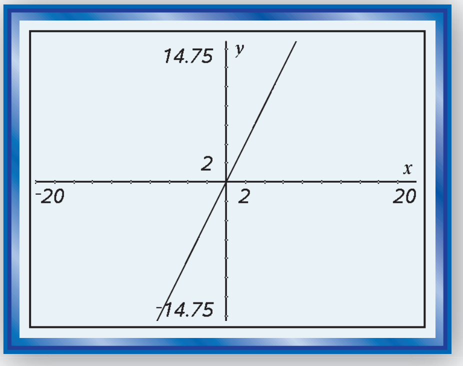

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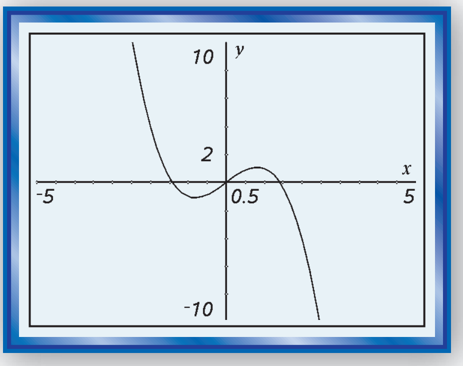

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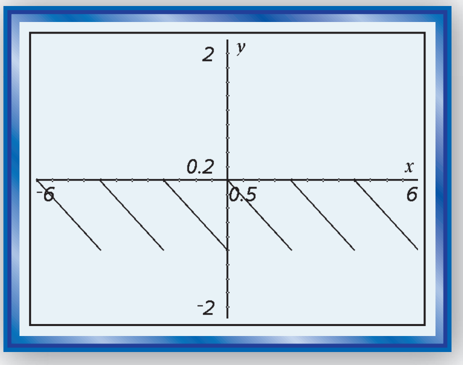


19. Below is a table of average monthly high temperatures T (all in degrees Fahrenheit) for Phoenix, Arizona.

Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
66	70	75	84	93	103	105	103	99	88	75	66

a. Explain why these data could be modeled by a sine wave.

b. Estimate the domain and range of a sinusoidal function that models these data.

20. a. Graph $y = \sin x$ and $y = \sin(180° - x)$.

b. What identity is suggested by the graphs?

21. a. Graph the function with equation $y = \sin x + \cos x$. Does this appear to be a periodic function? If so, what is its period?

b. What are the domain and range of this function?

REVIEW

22. **Multiple Choice** Which of the following is equal to sin (-45°)? **(Lesson 10-5)**

A sin 45° B sin 135°

C sin 405° D sin 675°

In 23 and 24, give the exact value without using a calculator. (Lesson 10-4)

23. cos 270° 24. tan 180°

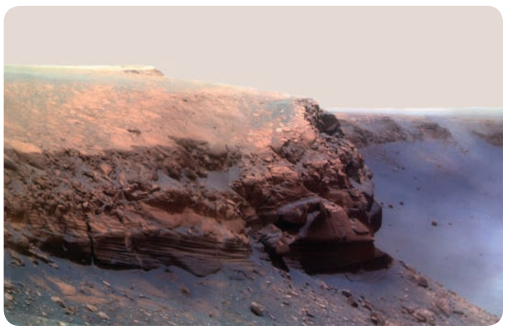
25. On Mars, the height h in meters of a thrown object at time t seconds is given by $h = -1.86t^2 + v_0t + h_0$. A space traveler standing on a 47-meter high Martian cliff tosses a rock straight up with an initial velocity of $15\frac{\text{m}}{\text{sec}}$. **(Lessons 6-7, 6-4)**

a. Write an equation to describe the height of the rock at time t.

b. Graph your equation in Part a.

c. What is the maximum height of the rock to the nearest meter?

d. To the nearest tenth of a second, when does the rock hit the Martian ground?



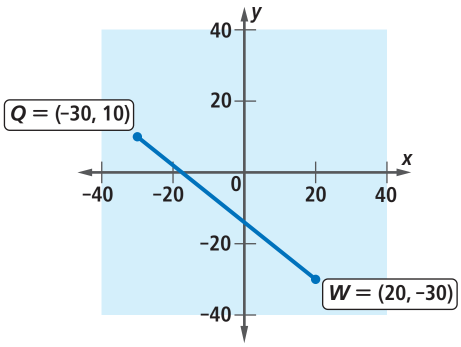
Cape Verde juts out from the walls of Victoria Crater on Mars.

In 26 and 27, $A = \begin{bmatrix} -72 & -27 \\ 8 & 3 \end{bmatrix}$.

26. a. Find det A.
    b. Does A^{-1} exist? If so, find it. If not, explain why it does not exist. **(Lesson 5-5)**

27. a. Find an equation for the line through the two points represented by matrix A.
    b. What kind of variation is described by the answer to Part a? **(Lessons 4-1, 3-4, 2-1)**

28. Approximate QW to the nearest hundredth. **(Lesson 4-4)**



EXPLORATION

29. Oscilloscopes can be used to display sound waves. Search the Internet to find some sites that simulate oscilloscope output for different sounds. Do additional research about sound waves and report on the following.
    a. the oscilloscope patterns for at least two sound waves (for example, whistling a tune, middle C on the piano)
    b. the meaning of frequency and amplitude of a sound wave
    c. the effect on sound tone as a result of changes in amplitude and frequency