Lesson
10-5 Relationships among Sines and Cosines

Vocabulary

identity

tangent of θ
(for all values of θ)

BIG IDEA Many properties of sines and cosines follow logically from the definition $(\cos \theta, \sin \theta) = R_\theta(1, 0)$ and properties of the unit circle.

With a calculator, it is easy to determine the values of cos θ and sin θ for any value of θ. But how can you check that you are correct? If θ is a multiple of 90°, you can find cos θ and sin θ by visualizing the exact location of $R_\theta(1, 0)$. If $0 < \theta < 90°$, then $R_\theta(1, 0)$ is in the first quadrant and you can estimate the values by drawing a right triangle. For other values of θ, you can use the symmetry of the unit circle.

Mental Math

Jamal is packing bulk auto parts into boxes to ship to automotive stores. He can only send full boxes. How many parts will Jamal have left over if he has

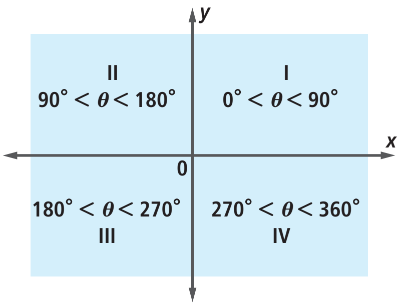
a. 672 windshield-wiper blades and 160 fit in a box?

b. 223 fan belts and 24 fit in a box?

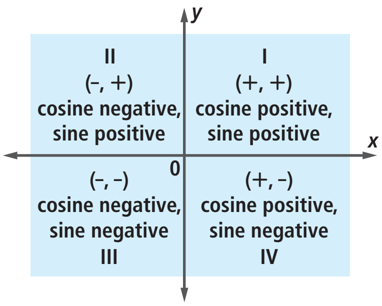
c. 17 fuel pumps and he can use boxes that fit 4 or 5 pumps?

Determining the Signs of cos θ and sin θ

When θ is not a multiple of 90°, (cos θ, sin θ) is in one of the four quadrants. As the figure below at the left shows, each quadrant is associated with one-fourth of the interval $0° < \theta < 360°$.



Location of $R_\theta(1, 0)$



Signs of cos θ and sin θ

The quadrants enable you to determine quickly whether cos θ and sin θ are positive or negative. Refer to the figure above at the right. Because cos θ is the first or x-coordinate of the image, it is positive when $R_\theta(1, 0)$ is in Quadrant I or IV and negative when $R_\theta(1, 0)$ is in Quadrant II or III. Because sin θ is the second or y-coordinate of $R_\theta(1, 0)$, sin θ is positive when $R_\theta(1, 0)$ is in Quadrant I or II and negative when $R_\theta(1, 0)$ is in Quadrant III or IV.

You do *not* need to memorize this information. You can always rely on the definition of cos θ and sin θ or visualize $R_\theta(1, 0)$ on the unit circle.

STOP **QY**

QY

A value of cos θ or sin θ is given. Identify the quadrant of $R_\theta(1, 0)$ and tell whether the value is positive or negative.

a. cos 212°

b. sin 212°

c. cos –17°

d. sin –17°

Determining sin θ from cos θ, and Vice Versa

Relationships that are true for all values of variables in a domain are called **identities**. The basic identity relating sin θ and cos θ comes from the Pythagorean Distance Formula. So it is called the *Pythagorean Identity*.

Pythagorean Identity Theorem

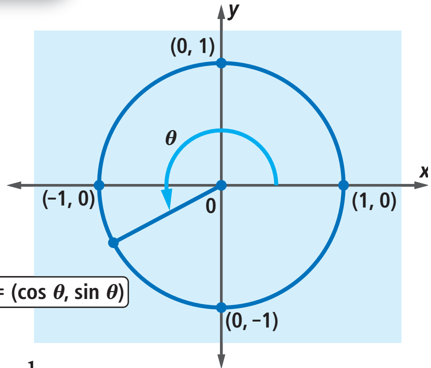
For all θ, $(\cos\theta)^2 + (\sin\theta)^2 = 1$.

Proof For any θ, $(\cos\theta, \sin\theta)$ is a point on the unit circle. Let $A = (\cos\theta, \sin\theta)$ and $O = (0, 0)$. By the Pythagorean Distance Formula,

$$OA = \sqrt{(\cos\theta - 0)^2 + (\sin\theta - 0)^2}.$$

But $OA = 1$ for every point A on the circle. Substituting 1 for OA and squaring both sides,

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$



To avoid having to write parentheses, $(\sin\theta)^n$ is written $\sin^n\theta$, and powers of the other trigonometric functions are written similarly. With this notation, the Pythagorean Identity is:

$$\text{For all } \theta,\ \cos^2\theta + \sin^2\theta = 1.$$

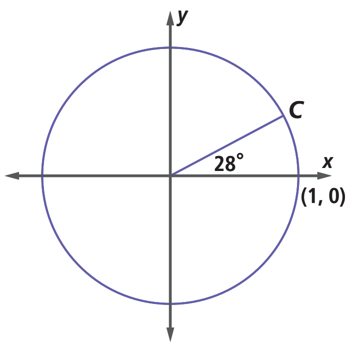
The Pythagorean Identity enables you to determine sin θ if cos θ is known, and vice versa. It also enables you to check sine and cosine values that you have obtained.

Example 1

Refer to the unit circle at the right.

a. Approximate the coordinates of point C to the nearest thousandth.

b. Verify that your coordinates in Part a satisfy the Pythagorean Identity.



Solution

a. Since $C = R_{28}(1, 0)$, $C = (\cos 28°, \sin 28°) \approx (0.883, 0.469)$.

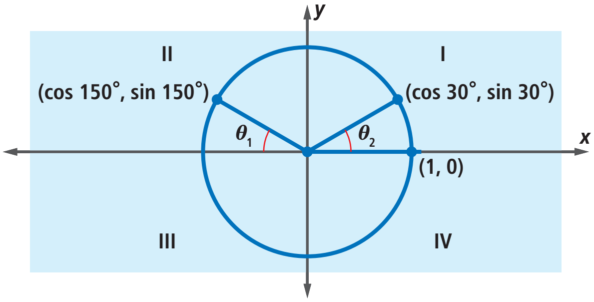
b. Here $\theta = 28°$.
$\cos^2 28° + \sin^2 28° \approx 0.883^2 + 0.469^2 = 0.99965 \approx 1$

Using Symmetry to Find Sines and Cosines

A circle is reflection-symmetric to any line through its center. For this reason, once you know the coordinates of one point on the circle, you can find the coordinates of many other points.

Second-Quadrant Values

When θ is between 90° and 180°, $R_\theta(1, 0)$ is in Quadrant II. Every point on the unit circle in Quadrant II is the image of a point on the circle in Quadrant I under a reflection over the *y*-axis. For instance, the point (cos 150°, sin 150°) is the reflection image over the *y*-axis of the point (cos 30°, sin 30°), which is in the first quadrant. Notice that the acute angles determined by these two points and the *x*-axis are congruent.



Recall that under r_y, the image of (x, y) is $(-x, y)$. The first coordinates of these points are opposites. Thus, $\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$.
The second coordinates are equal, so $\sin 150° = \sin 30° = \frac{1}{2}$.

Third-Quadrant Values

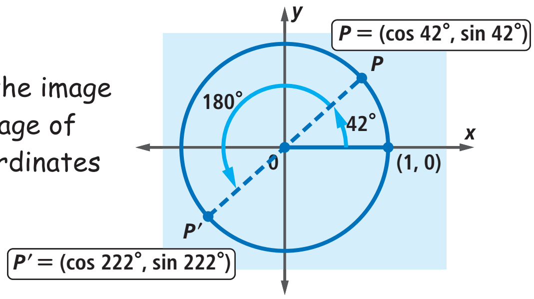
When a point is in Quadrant I, rotating it 180° gives a point in Quadrant III. Thus, to find sin θ or cos θ when $180° < \theta < 270°$, think of the angle with measure $\theta - 180°$.

Example 2

Show why sin 222° = –sin 42°.

Solution Make a sketch. P′ = (cos 222°, sin 222°) is the image of P = (cos 42°, sin 42°) under R_{180}. Because the image of (x, y) under a rotation of 180° is (–x, –y), P′ has coordinates (–cos 42°, –sin 42°). Thus, sin 222° = –sin 42°.

Check Using a calculator, sin 222° ≈ –0.669 and –sin 42° ≈ –0.669, so it checks.



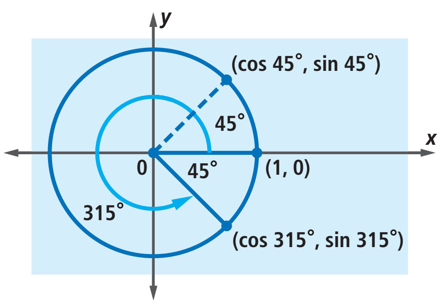
Fourth-Quadrant Values

Points in Quadrant IV are reflection images over the *x*-axis of points in Quadrant I.

Example 3

Find an exact value for cos 315°.

Solution cos 315° is the first coordinate of a point in Quadrant IV, so the cosine is positive. Reflect (cos 315°, sin 315°) over the x-axis. Since 360° – 315° = 45°, the image point is (cos 45°, sin 45°). Since the first coordinates of these points are equal, $\cos 315° = \cos 45° = \frac{\sqrt{2}}{2}$.



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Check A calculator shows $\cos 315° \approx 0.707$. Since 0.707 is an approximation for $\frac{\sqrt{2}}{2}$, it checks.

Activity

Step 1 Draw a good-sized copy of the figure of Example 1. Reflect C over the y-axis. Call the image C_2 (since it is in Quadrant II). Using the answer to Example 1, give the coordinates of C_2 to the nearest thousandth.

Step 2 Explain why R_{152} maps $(1, 0)$ onto C_2. Estimate $\cos 152°$ and $\sin 152°$ to the nearest thousandth.

Step 3 Reflect the original point C over the x-axis. Call the image C_4 (since it is in Quadrant IV). Using the answer to Example 1, give the coordinates of C_4 to the nearest thousandth.

Step 4 What is the magnitude θ of a rotation that maps $(1, 0)$ onto C_4? Use your answer to obtain $\sin \theta$ and $\cos \theta$ for this particular θ.

Step 5 Let $C_3 = R_{180}(C)$. Add C_3 to your figure and use it to obtain $\sin \theta$ and $\cos \theta$ for another value of θ.

In Lesson 10-4, you learned that if you know $\sin x$, you also know $\sin (x + n \cdot 360)$ because you can add or subtract multiples of 360° from the argument without changing the value of the sine. With the techniques in this lesson, if you know $\sin x$, you can use the Pythagorean Identity, reflections, and rotations to obtain the sines and cosines of many other arguments between 0° and 360°.

Questions

COVERING THE IDEAS

Fill in the Blank In 1–3, choose one of the following: *is always positive, is always negative,* or *may be positive or negative.*

1. If $R_\theta(1, 0)$ is in Quadrant II, then $\cos \theta$ __?__.
2. When $180° < \theta < 360°$, $\sin \theta$ __?__.
3. When $0° > \theta > -90°$, $\cos \theta$ __?__.
4. When $\cos \theta$ is negative, in which quadrant(s) can $R_\theta(1, 0)$ be?

In 5 and 6, a trigonometric value is given. Draw the corresponding point on the unit circle. Then, without using a calculator, state whether the value is positive or negative.

5. $\sin 271°$
6. $\cos 200°$

7. Suppose $\sin x = \frac{5}{13}$. Use the unit circle to
   a. find the two possible values of $\cos x$.
   b. explain why $\sin(-x) = -\frac{5}{13}$.

In 8 and 9, the given statement is true. Use the unit circle and transformations to explain why the statement is true. Then, verify the statement with a calculator.

8. $\cos 130° = \cos 230°$
9. $\sin 295° = -\sin 65°$

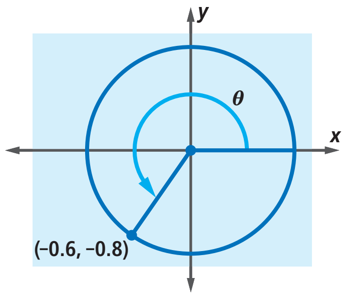
10. **Fill in the Blanks** Copy and complete with *positive* or *negative*. If $\angle B$ is obtuse, then $\cos B$ is __?__ and $\sin B$ is __?__.

APPLYING THE MATHEMATICS

In 11–14, find the exact value without a calculator.

11. $\cos 315°$
12. $\sin 135°$
13. $\sin(-120°)$
14. $\cos 930°$

15. Refer to the unit circle at the right. Find θ to the nearest degree.



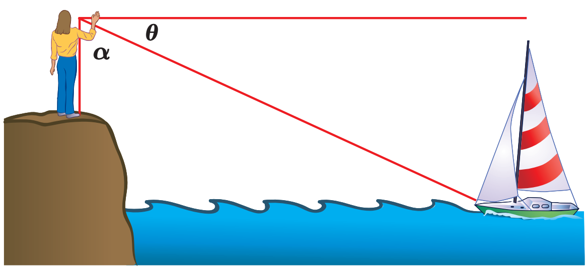
16. Given that $0° < x < 180°$ and $\cos x = -0.433$, find x to the nearest hundredth of a degree.

In 17–20, use this information. The tangent of θ, $\tan \theta$, is defined for all values of θ as $\frac{\sin \theta}{\cos \theta}$. This agrees with its right-triangle definition when $0° < \theta < 90°$.

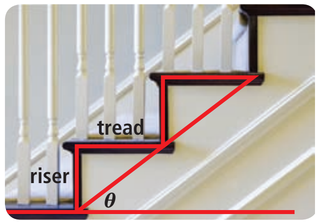
17. a. Use your calculator to evaluate $\sin 300°$, $\cos 300°$, and $\tan 300°$.
    b. Verify that $\frac{\sin 300°}{\cos 300°} = \tan 300°$.
18. What is the sign of $\tan \theta$ when $90° < \theta < 180°$? Justify your answer using the definition of $\tan \theta$.
19. Without a calculator, evaluate $\tan 900°$.
20. Without a calculator, give the exact value of $\tan 135°$.
21. Use the unit circle to explain why, for all θ, $\sin \theta = \sin(180° - \theta)$. (Hint: What is the image of a point when reflected over the y-axis?)

REVIEW

22. Without a calculator, evaluate $\sin 90°$ and $\cos 90°$. **(Lesson 10-4)**
23. **True or False** When measuring an object's distance using the parallax effect, a parallax angle of 110° is possible. **(Lesson 10-3)**
24. In the picture at the right, a person is standing on a cliff looking down at a boat. Which angle, θ or α, is the angle of depression? **(Lesson 10-2)**



25. Before 1992, one national building code specified that stairs in homes should be built with an $8\frac{1}{4}$-inch maximum riser height and 9-inch minimum tread. Falls are a leading cause of nonfatal injuries in the United States. In an effort to reduce the number of falls, the building code was changed in 1992 to require a 7-inch maximum riser and an 11-inch minimum tread. **(Lesson 10-2)**



a. By how many degrees did this change decrease the angle θ the stairs make with the horizontal?

b. Why do you think the writers of the code thought the new stairs would be safer?

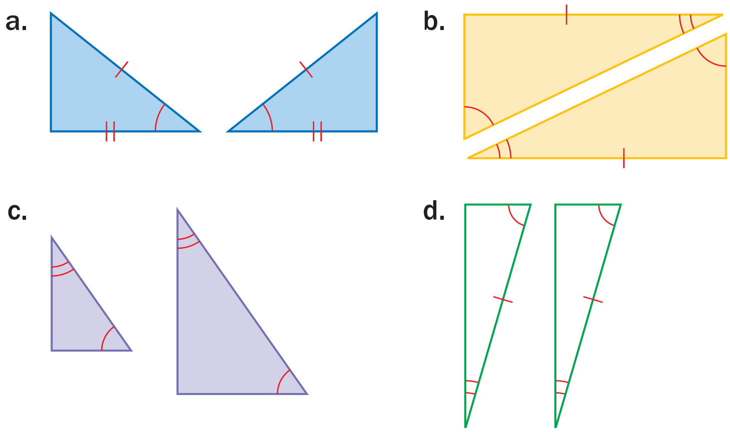
26. Cierra's grandmother put \$10,000 into a college account at Cierra's birth. The money is invested so she will have \$30,000 on her 18th birthday. **(Lessons 9-10, 9-3, 7-4)**

a. What rate of interest compounded annually will allow this?

b. If the interest is compounded continuously, what rate will be required?

27. Solve $3(x - 5)^6 = 12{,}288$ for x. **(Lesson 7-6)**

28. Determine whether the triangles in each pair are congruent. **(Previous Course)**



QY ANSWERS

a. III; negative

b. III; negative

c. IV; positive

d. IV; negative

EXPLORATION

29. **a.** Copy and complete the table at the right using a calculator or spreadsheet. Round answers to 6 decimal places.

b. You should find that $(\sin\theta - \tan\theta)$ gets closer and closer to 0. Explain why this happens.

θ	$\sin\theta$	$\tan\theta$	$\sin\theta - \tan\theta$
10°	?	?	?
5°	?	?	?
2°	?	?	?
1°	?	?	?
0.5°	?	?	?
0.1°	?	?	?