Lesson 10-4 The Unit-Circle Definition of Cosine and Sine

Vocabulary

unit circle

unit-circle definition of cosine and sine

BIG IDEA Every point P on the unit circle has coordinates of the form $(\cos\theta, \sin\theta)$, where θ is the magnitude of a rotation that maps (1, 0) onto P.

In a right triangle, the two angles other than the right angle each have a measure between 0° and 90°. So the definitions of sine, cosine, and tangent given in Lesson 10-1 only apply to measures between 0° and 90°. However, the sine, cosine, and tangent functions can be defined for all real numbers. To define cosines and sines for all real numbers, we use rotations with center (0, 0).

Mental Math

Let g be a geometric sequence with the formula $g_n = 120(0.75)^{n-1}$.

a. What is the second term of the sequence?

b. If the sequence models the height in inches of a dropped ball after the nth bounce, from what height in feet was the ball dropped?

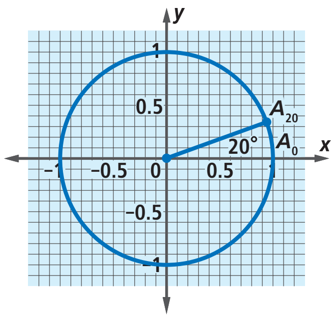
c. Could this sequence model the number of people who have heard a rumor?

Activity

MATERIALS compass, protractor, graph paper

Work with a partner.

Step 1 Draw a set of coordinate axes on a piece of graph paper. Let each side of a square on your coordinate grid have length 0.1 unit. With the origin as the center, use a compass to draw a circle with radius 1. Label the positive x-intercept of the circle as A_0. Your circle should look like the one below at the right.



Step 2 a. With a protractor, locate the image of $A_0 = (1, 0)$ under R_{20}. Label this point A_{20}.

b. Use the grid to estimate the x- and y-coordinates of A_{20}.

c. Use a calculator to find cos 20° and sin 20°.

Step 3 a. With a protractor, locate $R_{40}(1, 0)$. Label it A_{40}.

b. Use the grid to estimate the x- and y-coordinates of A_{40}.

c. Use a calculator to find cos 40° and sin 40°.

Step 4 a. Locate $R_{75}(1, 0)$. Label it A_{75}.

b. Estimate the x- and y-coordinates of this point.

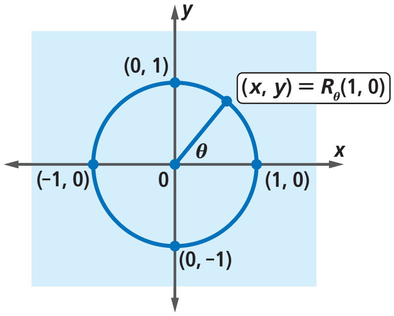
c. Use a calculator to evaluate cos 75° and sin 75°.

Step 5 **a.** Look back at your work for Steps 2–4. What relationship do you see between the x- and y-coordinates of $R_\theta(1, 0)$, $\cos \theta$, and $\sin \theta$?

b. Use your answer to Step 5 Part a to estimate the values of $\cos 61°$ and $\sin 61°$ from your figure without a calculator.

c. What is the relative error between your predictions in Step 5 Part b and the actual values of $\cos 61°$ and $\sin 61°$? Were you within 3% of the actual values?

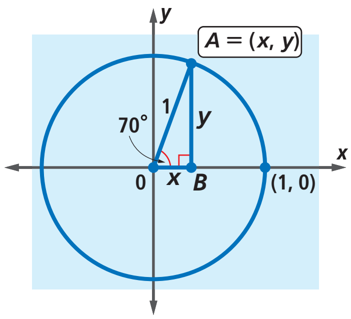
The Unit Circle, Sines, and Cosines



The circle you drew in the Activity is a *unit circle*. The **unit circle** is the circle with center at the origin and radius 1 unit. If the point (1, 0) on the circle is rotated around the origin with magnitude θ, then the image point (x, y) is also on the circle. The coordinates of the image point can be found using sines and cosines, as you should have discovered in the Activity.

Example 1

What are the coordinates of the image of (1, 0) under R_{70}?



Solution Let $A = (x, y) = R_{70}(1, 0)$. In the figure at the right, $OA = 1$ because the radius of the unit circle is 1. Draw the segment from A to $B = (x, 0)$. $\triangle ABO$ is a right triangle with legs of length x and y, and hypotenuse of length 1. Now use the definitions of sine and cosine.

$$\cos 70° = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

$$\sin 70° = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

The first coordinate is $\cos 70°$, and the second coordinate is $\sin 70°$. Thus, $(x, y) = (\cos 70°, \sin 70°) \approx (0.342, 0.940)$. That is, the image of (1, 0) under R_{70} is $(\cos 70°, \sin 70°)$, or about (0.342, 0.940).

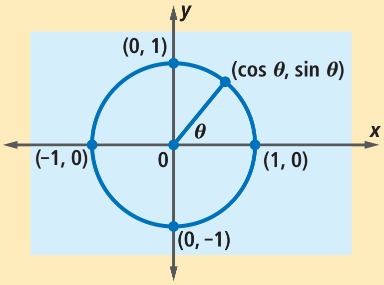
Check Use the Pythagorean Theorem with $\cos 70°$ and $\sin 70°$ as the lengths of the legs.

Is $(0.342)^2 + (0.940)^2 \approx 1^2$? $0.117 + 0.884 = 1.001 \approx 1$, so it checks.

The idea of Example 1 can be generalized to define the sine and cosine of any magnitude θ. Since any real number can be the magnitude of a rotation, this definition enlarges the domain of these trigonometric functions to be the set of all real numbers.

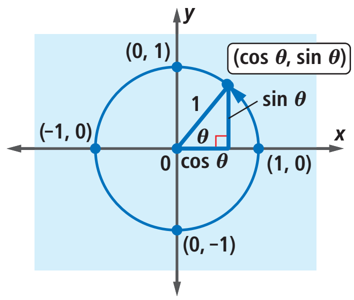
Unit-Circle Definition of Cosine and Sine

Let R_θ be the rotation with center (0, 0) and magnitude θ. Then, for any θ, the point ($\cos \theta$, $\sin \theta$) is the image of (1, 0) under R_θ.



Stated another way, $\cos \theta$ is the x-coordinate of $R_\theta(1, 0)$; $\sin \theta$ is the y-coordinate of $R_\theta(1, 0)$.

This unit-circle definition agrees with the right-triangle definition of cosine and sine for all magnitudes between 0° and 90°. Since the unit circle has radius 1, $\cos \theta$ is the ratio of the side adjacent to θ to the hypotenuse, and $\sin \theta$ is the ratio of the side opposite θ to the hypotenuse.



Cosines and Sines of Multiples of 90°

Sines and cosines of multiples of 90° can be found from the unit-circle definition without using a calculator.

Example 2

Explain how to use the unit circle to find

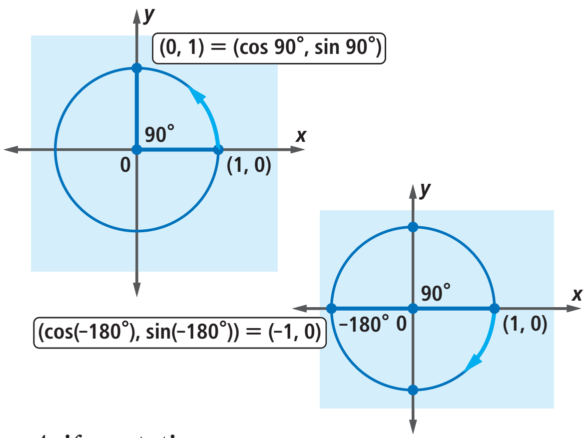
a. $\cos 90°$. **b.** $\sin(-180°)$.

Solution

a. Think: $\cos 90°$ is the x-coordinate of $R_{90}(1, 0)$.
$R_{90}(1, 0) = (0, 1)$. So $\cos 90° = 0$.

b. $R_{-180}(1, 0) = (-1, 0)$. Since $\sin(-180°)$ is the y-coordinate of this point, $\sin(-180°) = 0$.

Check Check these values on your calculator.



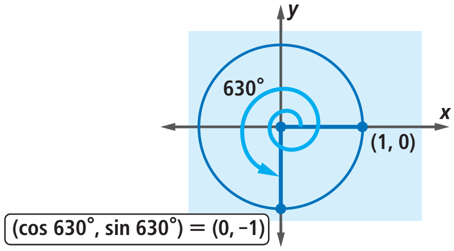
As you saw in your study of geometry and in Chapter 4, if a rotation of magnitude x is followed by a rotation of magnitude y with the same center, then the composite transformation is a rotation with magnitude $x + y$. That is, $R_x \circ R_y = R_{x+y}$. Furthermore, rotations of multiples of 360° are the identity transformation. Consequently, if you add or subtract integer multiples of 360° from the magnitude of a rotation, the rotation is the same. This means that you can add or subtract integer multiples of 360° from the arguments of the sine or cosine functions, and the value remains the same.

Example 3

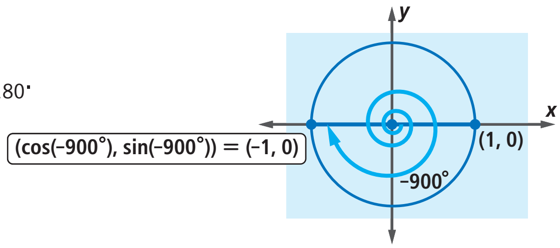
a. Find sin 630°.

b. Find cos –900°.

Solution Add or subtract multiples of 360° from the argument until you obtain a value from 0° to 360°.



a. $630° - 360° = 270°$. So R_{630} equals one complete revolution followed by a 270° rotation, and R_{630} and R_{270} have the same images. $R_{630}(1, 0) = R_{270}(1, 0) = (0, -1)$. So $\sin 630° = -1$.



b. Add $3 \cdot 360°$ to –900° to obtain a magnitude from 0° to 360°. $-900° + 3 \cdot 360° = 180°$. $R_{-900} = R_{180}$. So $\cos -900° = \cos 180° = -1$.

Questions

COVERING THE IDEAS

1. **Fill in the Blanks** If (1, 0) is rotated θ degrees around the origin,
   a. $\cos\theta$ is the __?__-coordinate of its image.
   b. $\sin\theta$ is the __?__-coordinate of its image.

2. **True or False** The image of (1, 0) under R_{23} is (sin 23°, cos 23°).

3. **Fill in the Blanks** $R_0(1, 0) =$ __?__, so cos 0° = __?__ and sin 0° = __?__.

4. Explain how to use the unit circle to find sin 180°.

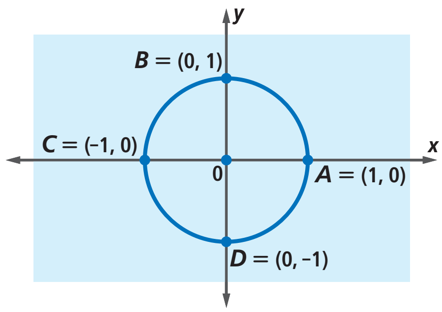
In 5–7, use the unit circle to find the value.

5. cos 90°
6. sin (–90°)
7. cos 270°

8. If (1, 0) is rotated –42° about the origin, what are the coordinates of its image, to the nearest thousandth?

9. **Fill in the Blanks**
   a. A rotation of 540° equals a rotation of 360° followed by __?__.
   b. The image of (1, 0) under R_{540} is __?__.
   c. Evaluate sin 540°.

In 10–12, suppose $A = (1, 0)$, $B = (0, 1)$, $C = (-1, 0)$, and $D = (0, -1)$. Which of these points is the image of (1, 0) under the stated rotation?



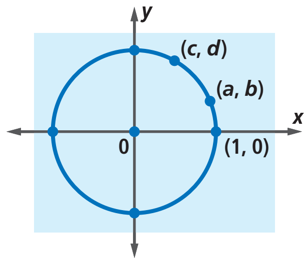
10. R_{450}
11. R_{540}
12. R_{-720}

In 13 and 14, evaluate without using a calculator.

13. cos 450° and sin 450° **14.** cos(-720°) and sin(-720°)

APPLYING THE MATHEMATICS

In 15–20, which letter on the figure at the right could stand for the indicated value of the trigonometric function?



15. cos 80° **16.** sin 80° **17.** cos(-280°)

18. sin 800° **19.** cos 380° **20.** sin(-340°)

In 21 and 22, find a solution to the equation between 0° and 360°. Then check your answer by using a calculator to approximate both sides of the equation to the nearest thousandth.

21. $\cos 392° = \cos x$ **22.** $\sin(-440°) = \sin y$

23. **a.** What is the largest possible value of $\cos \theta$?

b. What is the smallest possible value of $\sin \theta$?

In 24 and 25, verify by substitution that the statement holds for the given value of θ.

24. $(\cos \theta)^2 + (\sin \theta)^2 = 1; \theta = 7290°$

25. $\sin \theta = \sin(180° - \theta); \theta = -270°$

REVIEW

26. If an object has a parallax angle of 3° from two sites 100 meters apart, about how far away is the object? **(Lesson 10-3)**

27. Why must the observation sites be very far apart to determine the distance to a star by parallax? **(Lesson 10-3)**

28. A private plane flying at an altitude of 5000 feet begins its descent along a straight line to an airport 5 miles away. At what constant angle of depression does it need to descend? **(Lesson 10-2)**

29. A submarine commander took a sighting at sea level of the aircraft carrier USS Enterprise, the tallest ship in the U.S. Navy at 250 feet. He knew that the top of the ship was about 210 feet above sea level, and he noted that the angle of elevation to the top of the mast was 4°. How far from the Enterprise was the submarine? **(Lesson 10-1)**

30. Use the distance formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ to find the distance between (-2, 3) and (2, -7). **(Lesson 4-4)**

EXPLORATION

31. In Chapter 4, you used the Matrix Basis Theorem to develop rotation matrices for multiples of 90°. Use that theorem and the unit circle to produce a rotation matrix for any magnitude θ.