Lesson 10-3 Trigonometry, Earth, Moon, and Stars

Vocabulary

parallax angle

BIG IDEA Trigonometry has been used to estimate with great accuracy very large distances, such as are found in the solar system and in our galaxy.

Over 2000 years ago the ancient Greeks observed that when a ship arrived in port, the mast was the first part of the ship that could be seen. They concluded that Earth's surface must be curved. Using trigonometry and without modern equipment, the ancient Greeks were able to estimate the circumference of Earth and the distance from Earth to our Moon with remarkable accuracy. Their methods were so powerful that the same ideas are still used today to measure interstellar distances. One of these ideas is that of *parallax*.

Mental Math

Is the transformation an isometry, a similarity transformation, or neither?

a. $r_{y=x} \circ R_{200}$

b. $T_{2,16} \circ S_{2,4.5}$

c. $S_{3.5} \circ R_{450} \circ S_{\frac{2}{7}} \circ r_x$

Activity 1

Work with a partner.

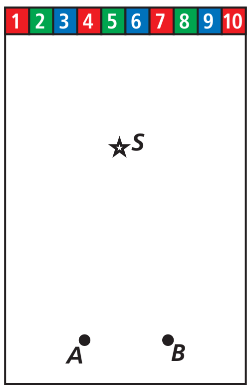
Step 1 Have your partner stand about eight feet in front of you. Extend your arm in front of your face and hold up one thumb. Close your left eye. Then open your left eye and close your right eye.

Your thumb and your partner both appear to jump to new locations. Which jumps more? Does the background behind your partner jump as well?

Take four steps back and repeat the experiment. What moves now?

Step 2 When you switched eyes, you changed the position from which you viewed your thumb and partner by about three inches. As a result, both your thumb and your partner appeared to move across the background behind them. This illusion of motion created by changing viewing positions is called *parallax.*

Copy the parallax diagram at the right. Draw lines from the observation sites *A* and *B* through the star *S* to the row of numbered squares in the background. The line from *A* indicates which square the star appears to obscure when viewed from the left site. That square marks the *apparent position* of the star from site *A*. Record that square, and the square obscured from site *B*.



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Step 3 As you noticed with your partner and your thumb, objects that are closer to the observer appear to move sideways more than objects that are far away.

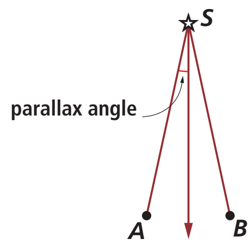
a. On the diagram you copied in Step 2, draw three more stars between the two observation sites and the shaded background.

b. Draw lines to identify each star's apparent position from each site and record the results.

c. For each pair of observations, determine how many squares the star "jumped." How does that number relate to the angle formed by site *A*, the star, and site *B*?

d. Write a sentence explaining why closer objects seem to jump sideways more than more distant objects.

In Activity 1 you measured the parallax effect by the magnitude of background movement. Another way is to measure $\angle ASB$. The larger the measure of this angle, the closer the star is to A and B. In practice, astronomers use half this angle and call it the **parallax angle**.



Astronomers measure parallax angles with special equipment, including satellites and telescopes with scales for measuring angles. You and a partner can simulate what they do by making a larger version of the parallax diagram used in Activity 1. One partner takes the place of the star, and the other makes observations from two sites.

Activity 2

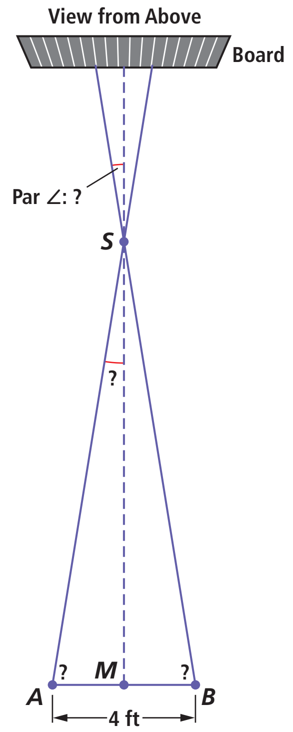
MATERIALS whiteboard or butcher paper, ruler, string

Step 1 **a.** Mark points *A* and *B* on the floor 4 feet apart so that $\overline{AB}$ is parallel to, and several feet from, a whiteboard or wall covered in butcher paper.

b. Draw equally spaced vertical lines on the board. Use a ruler to accurately space the lines 1 inch apart for every foot between $\overline{AB}$ and the board. (For example, if $\overline{AB}$ is 16 feet from the board, the lines should be 16 inches apart.)

Step 2 The star needs to be equidistant from the two observation sites. (In astronomical parallax, the variations in Earth's position are so small compared to the distances being measured that this condition is easy to meet.) Use string to construct the perpendicular bisector $\overleftrightarrow{MS}$ of $\overline{AB}$. Have your partner, the star, stand on $\overleftrightarrow{MS}$ between $\overline{AB}$ and the background. Observe your partner's head from *A* and from *B* by closing one eye and noting with the other eye the line on the board that appears to be closest to your partner's head. Close the same eye each time. Record the number of spaces between the observed board lines.

Step 3 Convert your range of spaces to an angle. If A and B are 16 feet from the board, each space corresponds to an angle of about 5°. So if you observed a 3.5-space difference from the two sites, this corresponds to a $5 \cdot 3.5 = 17.5°$ angle. Half this angle (8.25°) is the parallax angle shown in the diagram at the right. Copy the diagram and write your angle measurement in the "Par ∠" space. If you are not able to do the physical experiment, use the lines in the diagram shown here to estimate a parallax angle.



Step 4 $\triangle ASB$ is isosceles. Use this fact to fill in the measures of all other labeled angles.

Step 5 Use trigonometry to find MS.

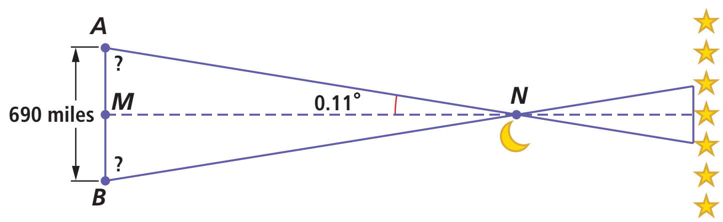
In Activity 2, MS is the distance from the observation sites line to the star. Using trigonometry, you were able to determine the distance to this remote object without having to go there and without sending anything to it.

The Distance to the Moon

Since ancient times, astronomers have used parallax to measure distances from Earth to celestial objects: the Moon, the planets, and in the last 200 years, nearby stars. In the second century BCE, Hipparchus used data recorded during a solar eclipse to estimate the distance from Earth to the Moon. Although the books describing his exact procedure have been lost, we know he used estimates of the Moon's parallax angle in his calculations. Now that you have seen how parallax works, you can use trigonometry to make your own estimates of the distance from Earth to the Moon.

Activity 3

Step 1 Data from observers of an eclipse in two Greek cities, Alexandria and Hellespont, led Hipparchus to approximate the Moon's parallax angle as 0.11°. This is the measure of $\angle ANM$ in the diagram at the right.



Because the Moon, represented by point N, is far away from Earth, $AN \approx BN$, so $\triangle ANB$ is isosceles with base $\overline{AB}$. Let M be the midpoint of $\overline{AB}$. Approximate the measures of the missing angles.

Step 2 Given that $AB = 690$ miles, use trigonometry to compute the distance MN from Earth to the Moon.

The actual distance from Earth to the Moon is about 238,000 miles. The data you used in Activity 3 leads to an underestimate by about 25%, which means that 0.11° is probably not the measure of the parallax angle when the Moon is exactly equidistant from the two cities. When Hipparchus did his calculations without modern trigonometry, he overestimated the distance by about 5%. The inaccuracy of Hipparchus' calculations may be due to the difficulty in verifying the data he worked with, and to the fact that none of the observers of his time had telescopes.

The Distance to a Star

To find distances to stars, you need to use observation sites that are much farther apart than 690 miles, and even then the parallax angles are extremely small. So while the ancient Greeks knew that the stars were extremely far away, they did not have an accurate way to measure those distances.

The first successful attempt to measure the distance to a star was by Friedrich Wilhelm Bessel in 1838, when he used the opposite sides of Earth's orbit around the Sun to measure a parallax angle to the star 61 Cygni. (This technique is called *annual parallax*.)

Friedrich Wilhelm Bessel has an asteroid named for him, 1552 Bessel.

Activity 4

The angle measured by Bessel was 0.314 arc-second; one arc-second is $\frac{1}{3600}$ degree. Earth's orbit is roughly circular with a diameter of 186,000,000 miles.

Step 1 Draw a diagram and compute the distance from Earth to 61 Cygni.

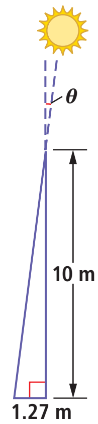
Step 2 One light-year is the distance light can travel in one Earth year, approximately $5.88 \cdot 10^{12}$ miles. How many light-years away is 61 Cygni?

The Circumference of Earth

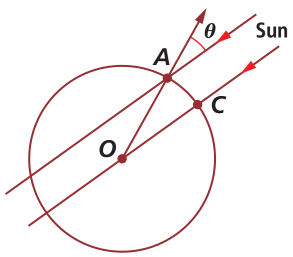
Hipparchus's original parallax computations gave the distance from Earth to the Moon in terms of Earth's radius. The radius of Earth was already known in Hipparchus's time, having been computed by Eratosthenes, a mathematician who lived in the third century BCE. Eratosthenes lived in Alexandria, Egypt, an important location of learning at the time. Eratosthenes found Earth's radius from its circumference. Activity 5 recreates his measurement of Earth's circumference.

Activity 5

Step 1 Eratosthenes knew that at noon on a particular day each year, in Cyene (an ancient city on the Nile River near what is now Aswan, Egypt), the Sun would shine directly overhead, even reflecting from the water in a deep well. In Alexandria on the same day each year, the Sun at noon was *not* directly overhead. In that city at noon, a 10-meter pole would cast a shadow approximately 1.27 meters long. Copy the diagram at the right, and use trigonometry to compute θ.



Step 2 Because the Sun is very far away from Earth, the rays reaching Earth are essentially parallel. If the Sun's rays strike Earth at different angles, it can only be because Earth's surface is curved. Copy the diagram at the right, substituting the angle measure θ you found in Step 1 and including the measures of all other angles in the figure. What is the degree measure of $\overset{\frown}{AC}$?



Step 3 Your work in Step 2 shows that the measure of $\overset{\frown}{AC}$ is $\frac{\theta}{360}$ of Earth's circumference. From caravans of merchants traveling between the two cities, Eratosthenes knew that Alexandria and Cyene were approximately 5,000 *stadia* apart. Although the exact correspondence is not known, scholars believe that 1 kilometer is approximately 6.27 stadia. Convert 5000 stadia into kilometers.

Step 4 If x is Earth's circumference and d is the distance between Alexandria and Cyene, then $\frac{\theta}{360}x = d$. Solve this equation for x using the values of θ and d that you computed in Steps 1 and 3.

The lack of modern technology at the time of Eratosthenes and Hipparchus makes their achievements even more amazing. Without using telescopes, much less calculators and computers, they were able to predict huge distances with incredible precision. The success of their techniques is a testament not just to their persistence but to the brilliance of their work and the power of mathematics in helping us understand the world around us.

Questions

COVERING THE IDEAS

1. What is meant by *parallax*?
2. If an object has a parallax angle of 6.5° from two sites 4 feet apart, about how many feet away is the object?
3. In recreating Hipparchus's computation of the Moon's distance from Earth, what did we assume about the two observation sites? What justifies this assumption?
4. Refer to Activity 4. The ancient Egyptians marked the seasons by the location of the brightest star in the night sky, Sirius. Modern observations show that Sirius has a parallax angle of 0.38 arc-second when measured from opposite ends of Earth's orbit.
   a. Compute the distance from Sirius to Earth in miles.
   b. Use your result from Part a to compute the distance between Earth and Sirius in light-years. (Sirius is one of the closest stars to Earth.)

Hipparchus of Rhodes is considered the Father of Astronomy and was the first person to systematically survey the sky.

APPLYING THE MATHEMATICS

5. Depending on conditions, an unaided human eye can distinguish objects as little as $\frac{1}{60}$ degree apart. For apparent motion, however, the human eye can detect only differences of about 1° or greater. Suppose that when you close one eye and open the other, an object appears to jump 1°, for a parallax angle of 0.5°. If your eyes are 3 inches apart, how many feet away is the object?
6. The nearest star to Earth is Proxima Centauri, 4.22 light-years away. Using the fact that 1 light-year $= 5.88 \times 10^{12}$ miles, compute the parallax angle you would observe for Proxima Centauri, using the endpoints of Earth's orbit as the two observation sites.

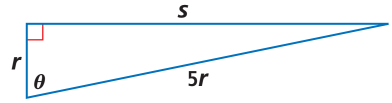
REVIEW

In 7 and 8, use the following information. The top of the Rock of Gibraltar is 426 meters above sea level. The Strait of Gibraltar, the body of water separating Gibraltar from the African continent, is 14 kilometers wide at its narrowest point. (Lessons 10-2, 10-1)

7. If the captain of the ship HMS Pinafore took a sighting on the Rock of Gibraltar and recorded a 14° angle of elevation, how far was the Pinafore from Gibraltar?
8. To avoid running into the African shore, what is the minimum angle of elevation possible?

9. In the triangle at the right, find each value. (Lesson 10-1)

a. $\sin \theta$ b. $\cos \theta$ c. $\tan \theta$



10. a. Solve the system $\begin{cases} y = x^2 - x - 2 \\ y = 3 \end{cases}$ by graphing.

b. Check your work using some other method. (Lessons 5-3, 5-2)

11. Find the coordinates of each point. (Lesson 4-8)

a. $R_{90}(1, 0)$ b. $R_{180}(1, 0)$

c. $R_{270}(1, 0)$ d. $R_{-90}(1, 0)$

12. After an initial 5000-mile break-in period, Mr. Euler changed the oil in his car and, thereafter changed the oil every 3000 miles. (Lessons 3-8, 3-6, 3-1)

a. What did the car's odometer read at the time of the second oil change?

b. What will the car's odometer read at the nth oil change after the break-in period?

c. How many times will Mr. Euler have changed his oil when the odometer reads 67,000?

13. State the Quadrants (I, II, III, or IV) in which (x, y) may be found if

a. x is negative and y is negative.

b. x is negative and y is positive.

c. $x = y$ and $xy \neq 0$. (Previous Course)

EXPLORATION

14. Let x be is the parallax angle for a star as seen from the endpoints of Earth's orbit, and let d be the star's distance from Earth in light-years.

a. Write a formula for d in terms of x.

b. Graph your equation from Part a using values of x between $0°$ and $\left(\frac{1}{3600}\right)^{\circ}$. Describe the graph clearly.

c. Using a table or a graph, compare the values of $f(x) = \tan x$ and $g(x) = \frac{\pi x}{180}$ for x between $0°$ and $\left(\frac{1}{3600}\right)^{\circ}$.

d. Use your observation in Part c to rewrite your formula in Part a. Does your rewritten formula hold for larger values of x, such as $20°$?