Lesson

10-2 More Right-Triangle Trigonometry

Vocabulary

inverse sine function, $\sin^{-1}$

inverse cosine function, $\cos^{-1}$

inverse tangent function, $\tan^{-1}$

angle of depression

BIG IDEA If you know two sides of a right triangle, you can use inverse trigonometric functions to find the measures of the acute angles.

In the last lesson, you used sines, cosines, and tangents of angles to find lengths of sides in right triangles. In this lesson, you will see how to use these ratios to find the measures of angles in right triangles.

Mental Math

Between which two consecutive whole numbers does each value fall?

a. log 15,823,556

b. $\ln e^{8.76}$

c. $\log_{18} 7$

d. $\log_5 620 + \log_9 77$

Finding an Angle from a Trigonometric Ratio

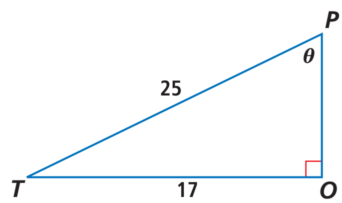
Given an angle measure x, you can use a calculator to find its sine, cosine, and tangent. That is, you can find y when $y = \sin x$, $y = \cos x$, or $y = \tan x$. Now, instead of knowing the angle, suppose you know its sine, cosine, or tangent. That is, suppose you know y in one of these situations. Can you find x?

The function that maps $\sin x$ onto x is the *inverse* of the function that maps x onto $\sin x$. Appropriately, this function is called the **inverse sine function**. On a calculator, this function is denoted by the symbol $\mathbf{sin^{-1}}$. Like the inverse of any function, when composed with the original function, the result is the identity. That is, for any angle θ between 0° and 90°, $\sin^{-1}(\sin\theta) = \theta$.

Example 1

Consider a right triangle *TOP* in which the hypotenuse $\overline{TP}$ has length 25 and $TO = 17$. What is $m\angle TPO$?

Solution Let θ be the unknown angle measure. Draw a figure, as shown at the right. From the figure, notice that $\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{17}{25} = 0.68$. To solve for θ, apply the inverse sine function to each side of the equation. Make sure that your calculator is in degree mode.



$$\sin\theta = 0.68$$

$$\sin^{-1}(\sin\theta) = \sin^{-1}(0.68)$$

$$\theta \approx 42.84°$$

$$m\angle TPO \approx 43°$$

The **inverse cosine function** (denoted **cos^{-1}**) and the **inverse tangent function (tan^{-1})** can be used in a similar way.

STOP **QY1**

> **QY1**
>
> Approximate the angle θ between 0° and 90° to the nearest degree.
>
> **a.** $\sin \theta = 0.324$
>
> **b.** $\tan \theta = 1.357$
>
> **c.** $\cos \theta = \frac{4}{7}$

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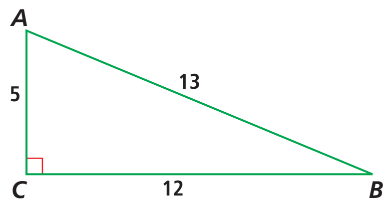
Example 2

Find the measures of the acute angles of the 5-12-13 right triangle below at the right.



Solution The two acute angles of this triangle are $\angle B$ and $\underline{\ ?\ }$.

Find the measure of $\angle B$ by using any of the trigonometric ratios.

$$\underline{\ ?\ }\,B = \frac{?}{?}$$

Apply the $\underline{\ ?\ }^{-1}$ function to both sides of the equation.

$$\underline{\ ?\ }^{-1}(\underline{\ ?\ }\,B) = \underline{\ ?\ }^{-1}\left(\frac{?}{?}\right)$$

Simplify. $\quad m\angle B \approx \underline{\ ?\ }°$

To find $m\angle A$ you could use another ratio, but applying the Triangle-Sum Theorem may be quicker.

$$m\angle A + m\angle B + m\angle C = 180°$$

$$m\angle A + \underline{\ ?\ } + 90° \approx 180°$$

$$m\angle A \approx \underline{\ ?\ }$$

Check Use another ratio to find $m\angle B$ and see if you get the same answers.

STOP **QY2**

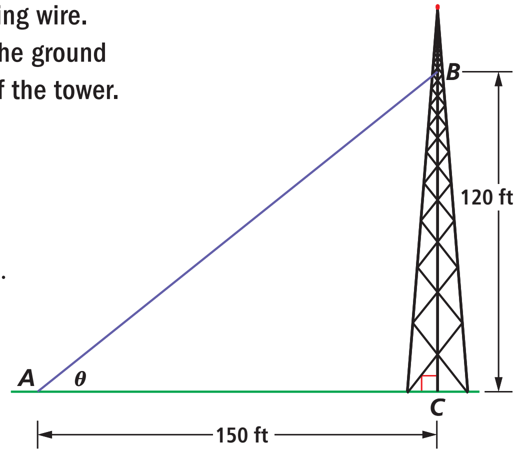
> **QY2**
>
> In Example 2, if you wanted to find $m\angle A$ using the cosine ratio, what fraction should be entered as the argument of $\cos^{-1}(\ )$?

Finding Angles of Elevation and Depression

One of the most important applications of trigonometry is to find distances and angle measures that are difficult or impossible to measure directly. In the last lesson, you used the trigonometric functions to find distances. The inverse trigonometric functions can be used to find angles.

Example 3

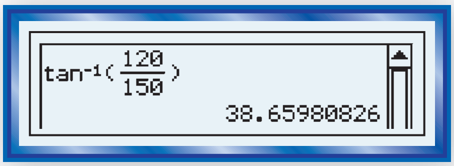
Suppose a cell tower is anchored to the ground by a supporting wire. The wire is attached at a point on the tower 120 feet above the ground and is attached to the level ground 150 feet from the base of the tower. What is the angle of elevation of the wire?



Solution In the drawing, $BC = 120$ feet, $AC = 150$ feet, and $\frac{BC}{AC} = \tan\theta$, where θ is the angle of elevation.

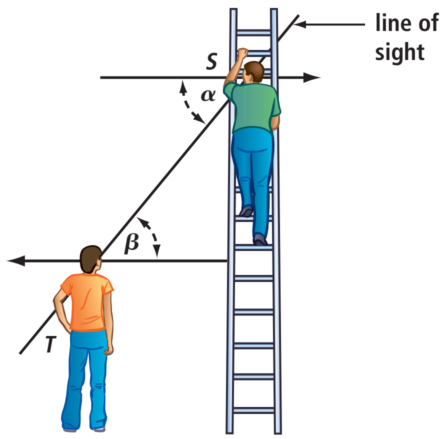
Then, $\tan\theta = \frac{120}{150} = 0.8$ and $\theta = \tan^{-1}(0.8) \approx 38.660$.

So, the angle of elevation of the wire is about 39°.



Check Does $\tan 39° = 0.8$? $\tan(39°) \approx 0.8098$.
Yes, it checks.

In the figure at the right, β (the Greek letter beta) represents the angle of elevation as person T looks up at person S. If S looks down at T, the angle α (the Greek letter alpha) between S's line of sight and the horizontal is called the **angle of depression**. The line of sight between T and S is a transversal for the parallel horizontal lines. Thus, β and α are alternate interior angles and are congruent. *So, the angle of elevation is equal to the angle of depression.*



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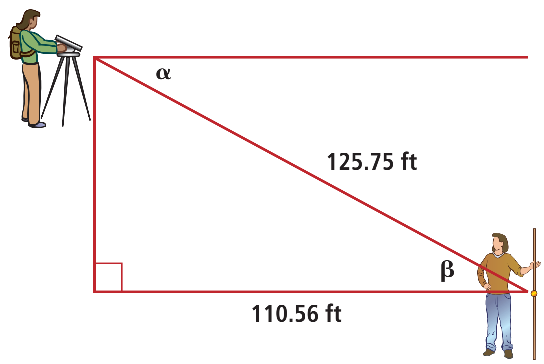
Example 4

A surveyor standing on a bridge points her scope towards an assistant standing on level ground 110.56 feet from the base of the bridge. Her surveying laser measures a direct (slant) distance to the assistant of 125.75 feet. To the nearest hundredth of a degree, find the angle of depression of the scope.



Solution The angle of depression α is not inside the triangle, so you cannot use it directly to set up a trigonometric ratio. However, the angle of depression α is equal to the angle of ___?___. So the angle of depression can be found using the ___?___ ratio.

$\cos \underline{\ ?\ } = \frac{?}{?}$ definition of cosine

$\cos^{-1}(\cos \underline{\ ?\ }) = \cos^{-1} \underline{\ ?\ }$ Apply the inverse cosine function.

$m\angle\beta \approx \underline{\ ?\ }$

Since $m\angle\alpha = m\angle\beta$, the angle of depression is about $\underline{\ ?\ }$ degrees.

Questions

COVERING THE IDEAS

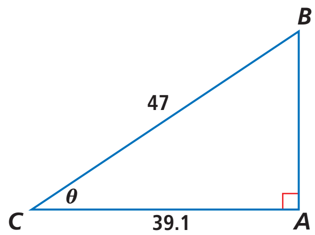
1. Write a key sequence for your calculator to find θ if $\sin\theta = 0.475$.

In 2 and 3, approximate to three decimal places.

2. $\cos^{-1}(0.443)$

3. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

4. Refer to $\triangle ABC$ at the right. Find θ to the nearest degree.



5. Refer to Example 2.
   a. Find $m\angle B$ using the tangent ratio.
   b. Find $m\angle A$ using the sine ratio.

6. Find the measures of the acute angles of an 8-15-17 right triangle to the nearest thousandth.

7. **Fill in the Blank** The angle of depression is the angle made between the line of sight to an object and the $\underline{\ ?\ }$.

8. Suppose a statue 15.5 meters high casts a shadow 21 meters long. What is the angle of elevation of the Sun?

9. Explain why the angle of depression from a point R to a point S equals the angle of elevation from S to R.

10. Refer to Example 4. Suppose the assistant stands 75 feet from the base of the bridge and the direct distance between the surveyor and the assistant is 120 feet. What is the angle of depression?

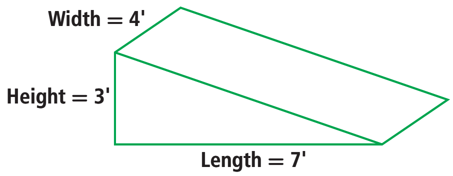
APPLYING THE MATHEMATICS

11. According to one railway company's guidelines, industrial railroad tracks must be built with a 1.5% grade or less. A 1.5% grade means that the track rises 1.5 feet vertically for every 100 feet horizontally.
    a. What is the slope of a line with a 1.5% grade?
    b. What is the tangent of the angle of elevation of a 1.5% grade?
    c. What angle does a track with a 1.5% grade make with the horizontal?

This train, on White Pass in Alaska, will rise approximately 3000 feet over just 20 miles.

12. Consider the line with equation $y = \frac{2}{3}x + b$.
    a. What is the slope of the line?
    b. What is the tangent of the acute angle that the line makes with x-axis?
    c. What is the measure of the acute angle that the line makes with the x-axis, to the nearest degree?
    d. What is the tangent of the acute angle that the line with equation $y = mx + b$ makes with the x-axis?

13. Haylee has the plans to build a skateboard launch ramp with dimensions as shown in the drawing at the right. What is the angle of elevation of the ramp?



14. Explain why the domain of the inverse sine function cannot include a number greater than 1.

15. Dawn Hillracer, an avid skier, rides 1600 meters on a chair lift to the top of a slope. If the lift has a 450-meter vertical rise, at what average angle of elevation does it ascend the slope?

REVIEW

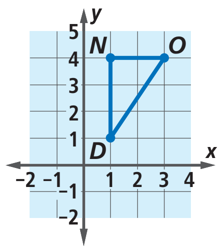
16. Chinese legend tells of a General Han Hsin, who used a kite during battle to work out the distance between his army and a castle so he could dig tunnels under the walls. Suppose the kite's string was 200 meters long and at an angle of 62° with the ground when the kite was above the courtyard of a castle. If the tunnels are dug straight, how long should the tunnels be in this case? Round to the nearest meter. **(Lesson 10-1)**

17. Rationalize the denominator of $\frac{11}{\sqrt{8}}$. **(Lesson 8-6)**

18. If an isosceles right triangle has a leg of length z, how long is its hypotenuse? **(Lesson 8-5, Previous Course)**

19. Solve for x: $rx^2 + sx + t = 0$. **(Lesson 6-7)**

In 20 and 21, use triangle *DON* at the right. (Lessons 4-7, 4-6, 4-4)



20. a. Find the coordinates of the vertices of $r_y(\triangle DON)$.
    b. Find the coordinates of the vertices of $r_x \circ r_y(\triangle DON)$.

21. Find DO.

EXPLORATION

22. a. Graph the $\sin^{-1}$ function over the domain $\{x \mid 0 < x < 1\}$. What is the range of the $\sin^{-1}$ function over this domain?
    b. Graph $\sin \circ \sin^{-1}$ over the domain $\{x \mid 0 < x < 1\}$. What is the range of this function over this domain?

QY ANSWERS

1. a. $\theta \approx 19°$
   b. $\theta \approx 54°$
   c. $\theta \approx 55°$
2. $\frac{5}{13}$