

## Lesson

## 10-1

## Three Trigonometric Functions

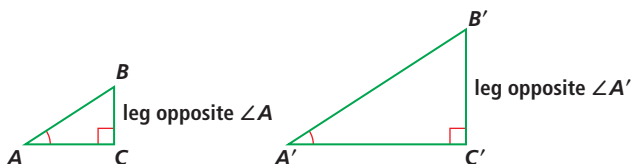
► **BIG IDEA** The sine, cosine, and tangent of an acute angle are each a ratio of particular sides of a right triangle with that acute angle.

Suppose a flagpole casts a 22-foot shadow when the Sun is at an angle of  $39^\circ$  with the ground. What is the height of the pole?

The height of the pole is determined by the given information because you know the measures of two angles of the right triangle (the  $39^\circ$  angle and the right angle) and the side they include.

The ASA Congruence Theorem indicates that all triangles with these measurements are congruent.

Problems like the one above led to the development of *trigonometry*. Consider the two right triangles  $ABC$  and  $A'B'C'$ , with  $\angle A \cong \angle A'$ .



By the AA Similarity Theorem, these triangles are similar, so ratios of the lengths of corresponding sides are equal. In particular,

$$\frac{B'C'}{BC} = \frac{A'B'}{AB}.$$

Exchanging the means produces an equivalent proportion.

$$\frac{B'C'}{A'B'} = \frac{BC}{AB}$$

Look more closely at these two ratios:

$$\frac{B'C'}{A'B'} = \frac{\text{length of the leg opposite } \angle A'}{\text{length of the hypotenuse of } \triangle A'B'C'}$$

and 
$$\frac{BC}{AB} = \frac{\text{length of the leg opposite } \angle A}{\text{length of the hypotenuse of } \triangle ABC}.$$

## Vocabulary

right-triangle definitions of sine (sin), cosine (cos), and tangent (tan)

sine function

cosine function

tangent function

angle of elevation

## Mental Math

Picture two similar triangles,  $\triangle BOW$  and  $\triangle TIE$ . Are the following ratios equal?

- $\frac{BO}{TI}$  and  $\frac{OW}{IE}$
- $\frac{WB}{ET}$  and  $\frac{TI}{BO}$
- $\frac{OB - OW}{WB}$  and  $\frac{TI - IE}{TE}$

Thus, in every right triangle with an angle congruent to  $\angle A$ , the ratio of the length of the leg opposite that angle to the length of the hypotenuse of the triangle is the same.

### Three Trigonometric Ratios

In similar right triangles, any other ratio of corresponding sides is also constant. These ratios are called *trigonometric ratios*. There are six possible trigonometric ratios. All six have special names, but three of them are more important and are defined here. The Greek letter  $\theta$  (theta) is customarily used to refer to an angle or to its measure.

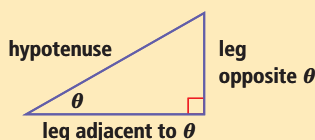
#### Right-Triangle Definitions of Sine, Cosine, and Tangent

In a right triangle with acute angle  $\theta$ ,

the **sine** of  $\theta = \frac{\text{length of leg opposite } \theta}{\text{length of hypotenuse}}$ ,

the **cosine** of  $\theta = \frac{\text{length of leg adjacent to } \theta}{\text{length of hypotenuse}}$ ;

the **tangent** of  $\theta = \frac{\text{length of leg opposite } \theta}{\text{length of leg adjacent to } \theta}$ .



Following a practice begun by Euler, we use the abbreviations  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  to stand for these ratios, and abbreviate them as shown below.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{opp.}}{\text{hyp.}}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{adj.}}{\text{hyp.}}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{opp.}}{\text{adj.}}$$



### Sine, Cosine, and Tangent Functions

The three correspondences that map an angle measure  $\theta$  in a right triangle onto each right-triangle ratio above define functions called the **sine**, **cosine**, and **tangent functions**.

$$\sin: \theta \rightarrow \sin \theta \qquad \cos: \theta \rightarrow \cos \theta \qquad \tan: \theta \rightarrow \tan \theta$$

From their right-triangle definitions, the domain of each of these functions is the set of all possible acute angle measures, that is,  $\{\theta \mid 0^\circ < \theta < 90^\circ\}$ . However, in a later lesson we will extend the domain so that  $\theta$  can have any degree measure, positive, negative, or zero.

#### READING MATH

Trigonometry, like algebra and geometry, is a branch of mathematics. The word *trigonometry* comes from the Greek words *trigon*, meaning “triangle” and *metron*, meaning “measure,” so trigonometry means “triangle measure,” that is, measuring the sides and angles of triangles.

#### QY

Write the ratios  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$  when  $\theta = \angle A$  in  $\triangle ABC$  on the previous page.

In the early history of trigonometry, mathematicians calculated values of the sine, cosine, and tangent functions and published the values in tables.

Today, people use calculators to find these values, with built-in programs using formulas derived from calculus. Most calculators allow you to enter angle measures in *degrees* or *radians*. You will learn about radians in Lesson 10-9. For now, make sure that any calculator you use is set to degree mode.

### Activity

**MATERIALS** ruler and protractor or DGS

See how close you can measure sides to obtain values of the trigonometric functions.

**Step 1** Draw a  $39^\circ$  angle and label it  $PQR$ .

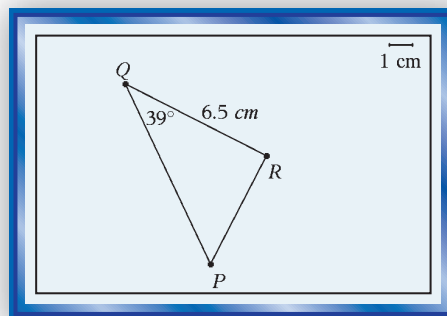
**Step 2** Draw a perpendicular from  $P$  to  $\overrightarrow{QR}$  to form a right triangle  $PQR$ , where  $\angle R$  is the right angle.

**Step 3** Measure the sides of  $\triangle PQR$  as accurately as you can.

**Step 4** Use the measures in Step 3 to calculate ratios to the nearest thousandth to estimate the sine, cosine, and tangent of  $39^\circ$ .

**Step 5** Set your calculator to degree mode. Find  $\sin 39^\circ$ ,  $\cos 39^\circ$ , and  $\tan 39^\circ$  to the nearest thousandth.

**Step 6** Subtract to calculate the error in each of the estimates you found by measuring. Divide each error by the calculator value to determine the *relative error* of the estimate. Consider yourself to have measured well if your relative error is less than 3%.



In this book, we usually give values of the trigonometric functions to the nearest thousandth. But when trigonometric values appear in long calculations, we do not round the calculator values until the end.

## Using Trigonometry to Find Sides of Right Triangles

### Example 1

Find the height of the flagpole mentioned in the first paragraph of this lesson.

**Solution** With respect to the  $39^\circ$  angle, the adjacent leg is known and the opposite leg is needed. Use the tangent ratio to set up an equation. Let  $x$  be the height of the flagpole.

$$\tan 39^\circ = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 39^\circ = \frac{x}{22}$$

Solve for  $x$ .

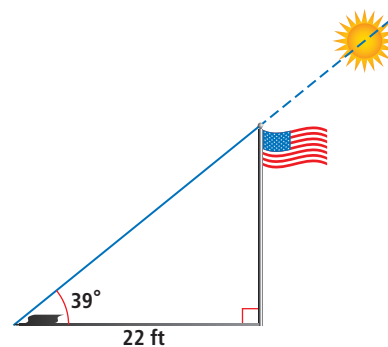
$$x = 22 \cdot \tan 39^\circ$$

From the Activity, we know that  $\tan 39^\circ \approx 0.810$ . Substitute.

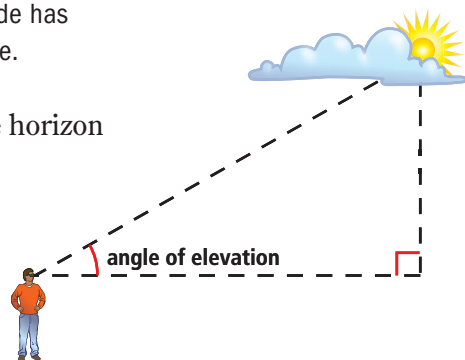
$$x \approx 22(0.810) = 17.82.$$

The flagpole is about 18 feet high.

**Check** Recall from geometry that within a triangle, longer sides are opposite larger angles. We have found that the side opposite the  $39^\circ$  angle is about 18 feet long. The angle opposite the 22-foot side has measure  $51^\circ$ , which is larger than  $39^\circ$ . So the answer makes sense.



The **angle of elevation** of an object, such as the Sun above the horizon or the peak of a mountain from its base, is the angle between the horizontal base of the object and the observer's line of sight to the object. If you know the angle of elevation of an object, you can use trigonometry to find distances that would otherwise be difficult to find.

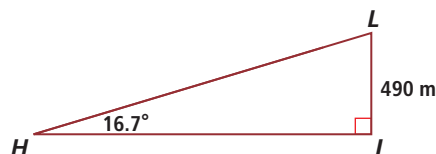


## GUIDED

### Example 2

The Zephyr Express chair lift in Winter Park, Colorado, has a vertical rise of about 490 meters. Suppose the lift travels at an average  $16.7^\circ$  angle of elevation. How many meters long is the ride?

**Solution** A diagram of this situation is given below.



You know the length of the side opposite the  $16.7^\circ$  angle and want to find the length of the hypotenuse  $LH$ . So use the sine ratio.

$$\sin \underline{\quad ? \quad} = \frac{\text{opp.}}{\text{hyp.}}$$

$$= \frac{?}{LH} \quad \text{Substitution}$$

$$LH = \frac{?}{?} \quad \text{Solve for } LH.$$

$$LH \approx \underline{\quad ? \quad} \quad \text{Calculate.}$$

So, the ride up the chair lift is about  $\underline{\quad ? \quad}$  meters long.

**Check** Use the Pythagorean Theorem to calculate  $HI$ .

$HI = \sqrt{\underline{\quad ? \quad}^2 - \underline{\quad ? \quad}^2} \approx \underline{\quad ? \quad}$  meters. Using the known angle, the cosine ratio relates  $HI$  and  $LH$ .  $\cos \underline{\quad ? \quad} \approx \underline{\quad ? \quad}$  and  $\frac{HI}{LH} \approx \underline{\quad ? \quad}$ . It checks.



A chair lift travels in a continuous loop, so riders must get on and off while it is moving. Many first-time riders fall on their way off.

## Drawing Auxiliary Lines to Create Right Triangles

When you wish to find the length of a segment and no right triangle is given, you can sometimes draw an auxiliary line to create a right triangle.

### Example 3

Each side in the regular pentagon *VIOLA* is 7.8 cm long. Find the length of diagonal  $\overline{VO}$ .

**Solution** Create a right triangle by drawing the segment perpendicular to  $\overline{VO}$  from *I*. Call the intersection point *P*, as in the drawing at the right. Recall that each angle in a regular pentagon has measure

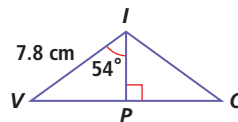
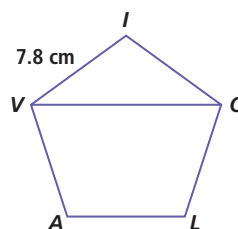
$$\frac{180(5-2)}{5} = 108^\circ. \text{ So } m\angle VIO = 108^\circ.$$

$\overline{IP}$  bisects  $\angle VIO$ . So  $m\angle VIP = 54^\circ$ . Since the hypotenuse of  $\triangle VIP$  is known and the opposite leg,  $VP$ , is needed, use the sine ratio.

$$\sin 54^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{VP}{7.8}$$

$$VP = 7.8 \cdot \sin 54^\circ \approx 6.3$$

The perpendicular  $\overline{IP}$  bisects  $\overline{VO}$ , so the diagonal is about  $2(6.3)$ , or 12.6 cm long.



## Using Special Right Triangles to Find Sines and Cosines

When  $\theta$  is  $30^\circ$ ,  $45^\circ$ , or  $60^\circ$ , you can use properties of 45-45-90 triangles or 30-60-90 triangles to find exact values of  $\sin \theta$  and  $\cos \theta$ .

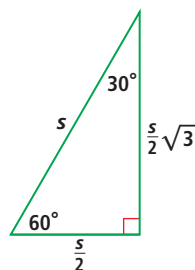
### Example 4

Use a 30-60-90 triangle to find  $\cos 30^\circ$ .

**Solution** Draw a 30-60-90 triangle. Label the  $30^\circ$  and  $60^\circ$  angles and let the length of the hypotenuse be  $s$ . Recall from geometry that since a 30-60-90 triangle is half of an equilateral triangle, if the length of the hypotenuse is  $s$ , the lengths of the legs are  $\frac{s}{2}$  and  $\frac{s}{2}\sqrt{3}$ .

So, the length of the leg opposite the  $30^\circ$  angle is  $\frac{s}{2}$ , and the length of the leg adjacent to the  $30^\circ$  angle is  $\frac{s}{2}\sqrt{3}$ .

Then, by the right-triangle definition of cosine,  $\cos 30^\circ = \frac{\frac{s}{2}\sqrt{3}}{s} = \frac{\sqrt{3}}{2}$ .



Using similar methods, you can find all the sine and cosine values below. You will use these special values of sine and cosine often in later mathematics courses.

$$\begin{array}{lll} \sin 30^\circ = \frac{1}{2} & \sin 60^\circ = \frac{\sqrt{3}}{2} & \sin 45^\circ = \frac{\sqrt{2}}{2} \\ \cos 30^\circ = \frac{\sqrt{3}}{2} & \cos 60^\circ = \frac{1}{2} & \cos 45^\circ = \frac{\sqrt{2}}{2} \end{array}$$

## Questions

### COVERING THE IDEAS

1. **Multiple Choice** The fact that different angles with the same measure have the same sine value is due to a property of which kinds of triangles?

A congruent triangles                      B isosceles triangles  
C right triangles                              D similar triangles

2. **Fill in the Blanks** Refer to  $\triangle JMS$  at the right.

a.  $\overline{JS}$  is the     ? of the triangle.

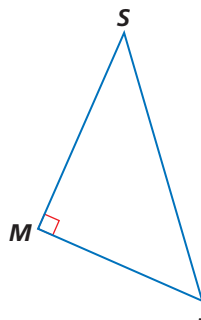
b.     ? is the leg opposite  $\angle J$ .

c.     ? is the leg adjacent to  $\angle J$ .

d.  $\frac{MJ}{JS} = \underline{\quad} J$

e.  $\frac{MS}{MJ} = \underline{\quad} J$

f.  $\frac{MS}{JS} = \underline{\quad} J$

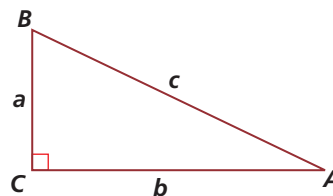


**Fill in the Blanks** In 3 and 4, use  $\triangle ABC$  at the right. Answer with expressions involving  $a$ ,  $b$ , and/or  $c$ .

3. a.  $\sin A = \underline{\quad}?$                       b.  $\cos A = \underline{\quad}?$                       c.  $\tan A = \underline{\quad}?$

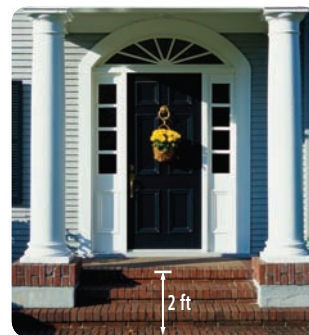
4. a.  $\sin B = \underline{\quad}?$                       b.  $\cos B = \underline{\quad}?$                       c.  $\tan B = \underline{\quad}?$

5. Suppose a construction crane casts a shadow 21 meters long when the Sun is  $78^\circ$  above the horizon. How high is the crane?
6. Suppose that a tree near your school casts a 32-foot shadow from its base when the Sun is  $72^\circ$  above the horizon. How tall is the tree?
7. Refer to Example 3. Use a trigonometric ratio to find  $IP$  to the nearest hundredth.
8. a. Use a 30-60-90 triangle to show that  $\cos 60^\circ = \frac{1}{2}$ .  
b. Use a 45-45-90 triangle to show that  $\sin 45^\circ = \frac{\sqrt{2}}{2}$ .

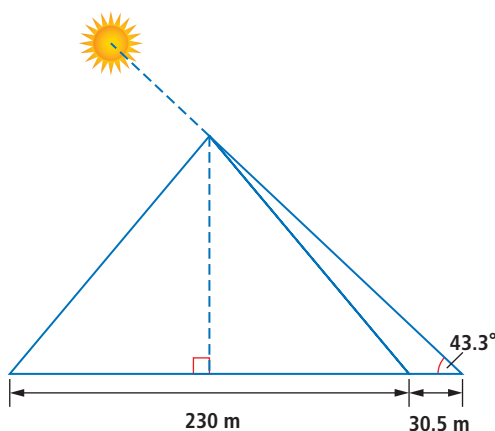


### APPLYING THE MATHEMATICS

9. The Americans with Disabilities Act specifies that the angle of elevation of ramps can be no greater than about  $4.76^\circ$ . The door pictured at the right is 2 feet off the ground.
- What is the length of the shortest ramp that will meet the code?
  - How far from the building will the ramp in Part a extend?
10. Explain why  $\tan 45^\circ = 1$ . (*Hint*: Draw a right triangle.)
11. a. Fill in the table.
- |                     |                       |
|---------------------|-----------------------|
| $\sin 30^\circ = ?$ | $\sin 40^\circ = ?$   |
| $\cos 60^\circ = ?$ | $\cos 50^\circ = ?$   |
| $\sin 15^\circ = ?$ | $\sin 9.3^\circ = ?$  |
| $\cos 75^\circ = ?$ | $\cos 80.7^\circ = ?$ |
- What is the relationship between the angles in each pair?
  - What is the relationship between the sine and cosine values you calculated in Part a?
  - Generalize your answers to Parts b and c as a conjecture.
  - Test your answer to Part d by finding sine and cosine of other angle pairs.



12. The Great Pyramid of Giza had a square base with side length 230 meters and a height of 146.6 meters when first built around 2570 BCE. Due to erosion, the pyramid's height has decreased, but its side lengths are still nearly the same. The pyramid casts a 30.5-meter shadow when the Sun is at a  $43.3^\circ$  angle of elevation, as shown below. What is the current height of the pyramid to the nearest meter?

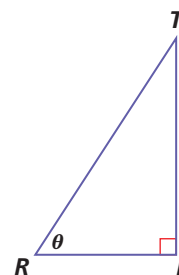


13. A regular octagon has diagonals of three different lengths. What is the length of the shortest diagonal if a side of the regular octagon has length 1 unit?

14. In  $\triangle TRI$  at the right,  $m\angle I = 90^\circ$ . Let  $m\angle R = \theta$ .

a. Find  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$ .

b. Use your answer from Part a to prove that  $\frac{\sin \theta}{\cos \theta} = \tan \theta$ .

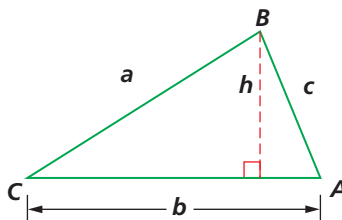


15. Use the figure at the right.

a. Prove that the area of  $\triangle ABC$  is

$$\frac{1}{2}ab \sin C. \text{ (Hint: Find } h\text{.)}$$

b. Find the area of  $\triangle ABC$  if  $m\angle C = 20.3^\circ$ ,  $a = 72$ , and  $b = 75$ .



16. If it takes 34.66 years for an investment to double under continuous compounding, how long would it take to be multiplied by a factor of 2.5? (Lessons 9-10, 9-3)

17. Solve for  $z$ :  $9\sqrt[3]{z-5} = 27$ . (Lesson 8-8)

18. Write two different expressions equal to  $\frac{x^2}{x\sqrt{8}}$ ,  $x \neq 0$ . At least one expression should have a rational denominator. (Lesson 8-6)

19. Suppose  $h$  is a function with an inverse. Simplify. (Lesson 8-3)

a.  $h(h^{-1}(\pi))$

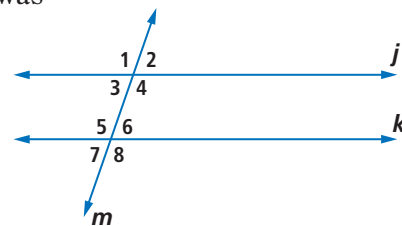
b.  $h^{-1}(h(\pi))$

20. One group of students ordered 5 hamburgers and 5 veggie burgers and paid \$66.00. Another group ordered 4 hamburgers and 7 veggie burgers and paid \$71.55. At these prices, what was the cost of one hamburger? (Lesson 5-4)

**True or False.** In 21 and 22, refer to the figure at the right where  $j \parallel k$ . Explain your answer. (Previous Course)

21.  $\angle 1 \cong \angle 7$

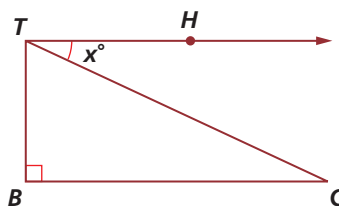
22.  $\angle 2 \cong \angle 6$



23. In the figure at the right,  $m\angle HTC = x^\circ$  and  $\overline{TH} \parallel \overline{BC}$ . (Previous Course)

a. Find  $m\angle BTC$ .

b. Find  $m\angle C$ .



### EXPLORATION

24. a. As  $\theta$  increases from  $0^\circ$  to  $90^\circ$ , what happens to the value of  $\sin \theta$ ? Does it increase? Does it decrease? Does it shift back and forth? Use values obtained from a calculator to find the pattern. Then use a geometrical argument and the definition of  $\sin \theta$  to explain why  $\sin \theta$  acts as it does.
- b. What happens to the value of  $\cos \theta$  over the same interval?
- c. What happens to the value of  $\tan \theta$ ?

### QY ANSWER

$$\sin \theta = \frac{BC}{AB}, \cos \theta = \frac{AC}{AB},$$

$$\tan \theta = \frac{BC}{AC}.$$