

## Lesson

## 9-6

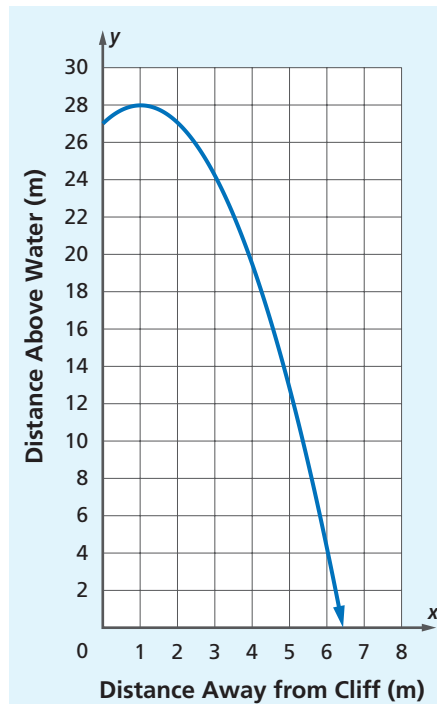
## Analyzing Solutions to Quadratic Equations

## Vocabulary

discriminant

**► BIG IDEA** The value of the discriminant  $b^2 - 4ac$  of a quadratic equation  $ax^2 + bx + c = 0$  can tell you whether the equation has 0, 1, or 2 real solutions.

In Acapulco, Mexico, cliff divers dive from a place called La Quebrada (“the break in the rocks”) 27 meters above the water. As you have learned, a diver’s path is part of a parabola that can be described using a quadratic equation. An equation that relates the distance  $x$  (in meters) away from the cliff and the distance  $y$  (in meters) above the water is  $y = -x^2 + 2x + 27$ . The graph at the right shows that when a diver pushes off the cliff, the diver arches upward and then descends.



Will the diver’s height reach 27.75 meters? 28 meters? 30 meters? You can use the equation, the graph, or the table to determine whether or not a diver reaches a particular height.

## Using a Graph or Table to Determine Solutions to a Quadratic Equation

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## Example 1

Consider the situation of a La Quebrada diver. Graph and generate a table of the parabola with equation  $y = -x^2 + 2x + 27$  to determine whether the diver will ever reach

- a. 27.75 meters.    b. 28 meters.    c. 30 meters.

## Mental Math

Solve the equation.

- a.  $a + 14 = 29$   
 b.  $b^2 + 14 = 39$   
 c.  $(c - 6)^2 + 14 = 39$



Adele Laurent of Denver, Colorado, dives from the La Quebrada cliff during the International Cliff Diving Championships in 1996, the first year that women were allowed to participate in the championships.

Source: Associated Press

**Solutions**

- a. A graph of  $y = -x^2 + 2x + 27$  is shown on the right using the window  $0 \leq x \leq 4$  and  $25 \leq y \leq 30$ . Also graphed is the line  $y = 27.75$ . This line crosses the parabola twice.

The diver reaches 27.75 meters twice, once on the way up and once on the way down.

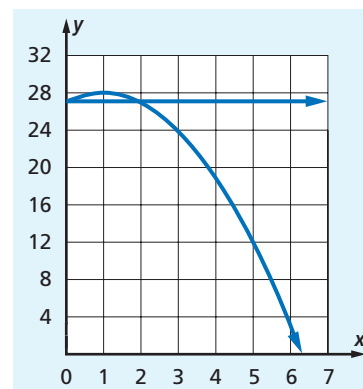
A table of  $y = -x^2 + 2x + 27$  is shown below. From the table it is evident that there are two distances  $x$  when  $y = 27.75$  meters: once 0.5 meter from the cliff and again 1.5 meters away.

- b. Suppose you draw the line  $y = 28$  on the graph. How many times does the line appear to intersect the graph?   ?

Now look at the table. It appears the diver reaches the height of 28 meters   ?   time(s). The diver reaches the height of 28 meters   ?   meter(s) from the cliff.

- c. Suppose you draw the line  $y = 30$  on the graph. How many times does the line appear to intersect the graph?   ?

Now look at the table below. It appears the diver reaches the height of 30 meters   ?   time(s).



X	Y1
0	27
0.5	27.75
1	28
1.5	27.75
2	27
3	18
4	7
5	-4
6	-15
7	-26

## Using the Quadratic Formula to Find the Number of Real Solutions

We can answer the same questions about the height of the diver using the Quadratic Formula.

### Example 2

Will the diver ever reach a height of

- a. 27.75 meters?    b. 28 meters?    c. 30 meters?

**Solutions**

- a. Let  $y = 27.75$  in the equation  $y = -x^2 + 2x + 27$ .

$$27.75 = -x^2 + 2x + 27$$

Add  $-27.75$  to both sides to put the equation in standard form.

$$0 = -x^2 + 2x - 0.75$$

(continued on next page)

Let  $a = -1$ ,  $b = 2$ , and  $c = -0.75$  in the Quadratic Formula. Then

$$x = \frac{-2 \pm \sqrt{2^2 - 4(-1)(-0.75)}}{2(-1)}$$

$$x = \frac{-2 + \sqrt{1}}{-2} \quad \text{or} \quad x = \frac{-2 - \sqrt{1}}{-2}$$

$$x = 0.5 \quad \text{or} \quad x = 1.5$$

The diver reaches a height of 27.75 meters twice, first at 0.5 meter from the cliff and second at 1.5 meters from the cliff.

b. Let  $y = 28$  in the equation  $y = -x^2 + 2x + 27$ .

$$28 = -x^2 + 2x + 27$$

Add  $-28$  to both sides to place the equation in standard form.

$$0 = -x^2 + 2x - 1$$

Let  $a = -1$ ,  $b = 2$ , and  $c = -1$  in the Quadratic Formula.

$$x = \frac{-2 \pm \sqrt{2^2 - 4(-1)(-1)}}{2(-1)} = \frac{-2 \pm 0}{-2} = 1$$

So the diver reaches 28 meters just once, 1 meter up from the cliff.

This agrees with the graph that shows the vertex to be  $(1, 28)$ .

c. Let  $y = 30$  in the equation  $y = -x^2 + 2x + 27$ .

$$30 = -x^2 + 2x + 27$$

Add  $-30$  to both sides to place the equation in standard form

$$0 = -x^2 + 2x - 3$$

Let  $a = -1$ ,  $b = 2$ , and  $c = -3$  in the Quadratic Formula. Then

$$x = \frac{-2 \pm \sqrt{2^2 - 4(-1)(-3)}}{2(-1)} = \frac{-2 \pm \sqrt{-8}}{-2}$$

Because no real number multiplied by itself equals  $-8$ , there is no square root of  $-8$  in the real number system. In fact, no negative number has a square root in the real number system. So  $30 = -x^2 + 2x + 27$  does not have a real number solution. This means that the diver never reaches a height of 30 meters. This is consistent with the graph that shows there is no point on the parabola with a height of 30 meters.

## The Discriminant of a Quadratic Equation

Look closely at the number under the square root in each of the solutions in Example 2. Notice that the number of solutions to a quadratic equation is related to this number, which is the number  $b^2 - 4ac$  in the Quadratic Formula. In the solution to Part a,  $b^2 - 4ac = 1$ , which is positive. Adding  $\sqrt{1}$  and subtracting  $\sqrt{1}$  results in two solutions. In the solution to Part b,  $b^2 - 4ac = 0$ , and adding  $\sqrt{0}$  and subtracting  $\sqrt{0}$ , yields the same result, 1. That quadratic equation has just one solution. There is no solution to the equation in Part c because  $b^2 - 4ac = -8$ , and  $-8$  does not have a square root in the set of real numbers.

Because the value of  $b^2 - 4ac$  *discriminates* among the various possible number of real number solutions to a specific quadratic equation, it is called the **discriminant** of the equation  $ax^2 + bx + c = 0$ . Stated below are the specific properties of the discriminant.

### Discriminant Property

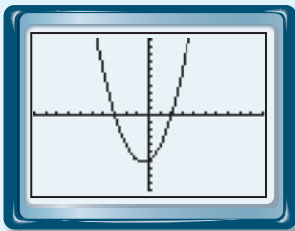
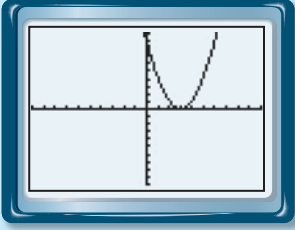
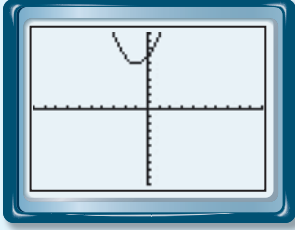
If  $ax^2 + bx + c = 0$  and  $a$ ,  $b$ , and  $c$  are real numbers ( $a \neq 0$ ), then:

When  $b^2 - 4ac > 0$ , the equation has exactly two real solutions.

When  $b^2 - 4ac = 0$ , the equation has exactly one real solution.

When  $b^2 - 4ac < 0$ , the equation has no real solutions.

An important use of the discriminant relates solutions of a quadratic equation to the  $x$ -intercepts of the related function. Specifically, the solutions to  $ax^2 + bx + c = 0$  are the  $x$ -intercepts of  $y = ax^2 + bx + c$ . So the discriminant tells you how many times the function  $f(x) = ax^2 + bx + c$  crosses the  $x$ -axis.

Quadratic Function	Value of $b^2 - 4ac$	Number of $x$ -intercepts	Graph (All screens are shown in the standard viewing window.)
$y = x^2 + x - 6$	$1^2 - 4(1)(-6) = 25$ positive	two	
$y = x^2 - 6x + 9$	$(-6)^2 - 4(1)(9) = 0$ zero	one	
$y = x^2 + 2x + 7$	$2^2 - 4(1)(7) = -24$ negative	zero	



#### ► QY

Determine the number of real solutions to  $6x^2 + 3x = -7$ .

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**Example 3**

How many times does the graph of  $y = 2x^2 + 16x + 32$  intersect the  $x$ -axis?

**Solution** Find the value of the discriminant  $b^2 - 4ac$ . Here  $a = \underline{\quad ? \quad}$ ,  $b = \underline{\quad ? \quad}$ , and  $c = \underline{\quad ? \quad}$ .

So  $b^2 - 4ac = \underline{\quad ? \quad}$ .

$= \underline{\quad ? \quad} = 0$

Because the discriminant is  $\underline{\quad ? \quad}$ , the graph of  $y = 2x^2 + 16x + 32$  intersects the  $x$ -axis  $\underline{\quad ? \quad}$  time(s).

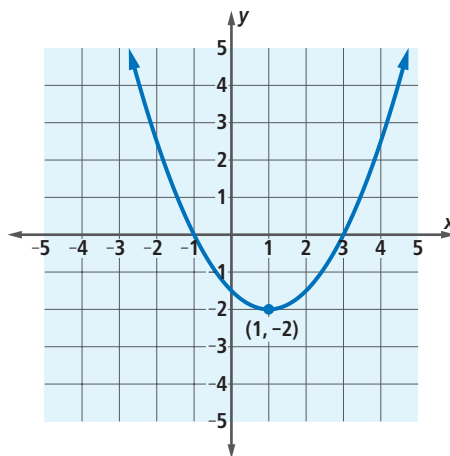
**Check** You should check your answer by graphing the equation with a calculator.

**Questions**

## COVERING THE IDEAS

In 1 and 2, refer to the La Quebrada cliff diver equation  $y = -x^2 + 2x + 27$  from Example 1.

- What equation can be solved to determine how far away (horizontally) from the cliff the diver will be when the diver is 27 meters above the water?
  - Will the diver reach a height of 27 meters above the water? If so, how many times?
- How far from the cliff will the diver be at 10 meters above the water?
- How many real solutions does a quadratic equation have when the discriminant is
  - negative?
  - zero?
  - positive?
- The discriminant of the equation  $ax^2 + bx + c = 0$  is  $-1,200$ . What does this indicate about the graph of  $y = ax^2 + bx + c$ ?
- The equation  $y = \frac{1}{2}x^2 - x - \frac{3}{2}$  is graphed at the right. Use the graph to determine the number of real solutions to each equation.
  - $\frac{1}{2}x^2 - x - \frac{3}{2} = -2$
  - $\frac{1}{2}x^2 - x - \frac{3}{2} = -3$
  - $\frac{1}{2}x^2 - x - \frac{3}{2} = 1$
  - $\frac{1}{2}x^2 - x - \frac{3}{2} = 5,000$



In 6–9, a quadratic equation is given.

- Calculate the value of the discriminant.
- Give the number of real solutions.
- Find all the real solutions.

6.  $2x^2 + x + 3 = 0$

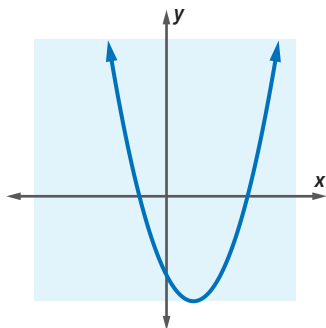
7.  $-4n^2 + 56n - 196 = 0$

8.  $22q^2 = q + 3$

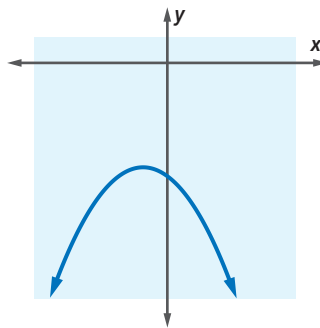
9.  $x = \frac{x^2}{6} + \frac{1}{4}$

In 10 and 11, an equation of the form  $y = ax^2 + bx + c$  is graphed. Tell whether the value of  $b^2 - 4ac$  is *positive*, *negative*, or *zero*.

10.



11.



In 12 and 13, a quadratic function  $f$  is described.

- Calculate the value of the discriminant of the quadratic equation  $f(x) = 0$ .
- Give the number of  $x$ -intercepts of the graph of  $f$ .

12.  $f(x) = 5x^2 + 20x + 20$

13.  $f(x) = 2x^2 + x - 3$

### APPLYING THE MATHEMATICS

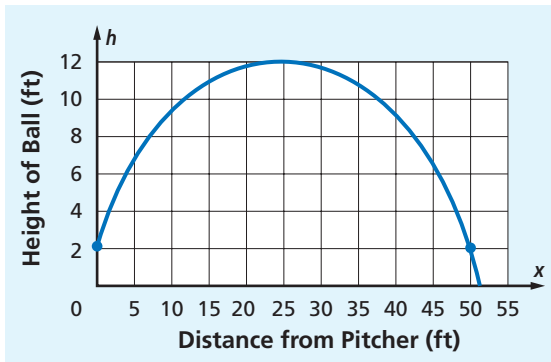
- For what value of  $h$  does  $x^2 + 6x + h = 0$  have exactly one solution?
- If the discriminant for the equation  $2x^2 + 4x + c = 0$  is 8, what is the value of  $c$ ?
- In Lesson 9-5, a diver's height  $h(t)$  above the water after  $t$  seconds was given by  $h(t) = -4.9t^2 + 4.3t + 10$ . Use the discriminant to find the time  $t$  when the diver reached the maximum height.
- Can any parabolas with an equation of the form  $y = ax^2 + bx + c$  *not* have a  $y$ -intercept? Why or why not?
- By letting  $x = m + 3$ , solve  $4(m + 3)^2 - 13(m + 3) - 35 = 0$  for  $m$ .

### REVIEW

- Solve  $45x^2 - 100 = 0$ . (Lesson 9-5)



In 20 and 21, use the following information. A softball pitcher tosses a ball to a catcher 50 feet away. The height  $h$  (in feet) of the ball when it is  $x$  feet from the pitcher is given by the equation  $h = -0.016x^2 + 0.8x + 2$ . (Lesson 9-4)



20. How high is the ball at its peak?
21. a. If the batter is 2 feet in front of the catcher, how far is the batter from the pitcher?  
b. How high is the ball when it reaches the batter?

In 22 and 23, state whether the parabola described by the equation opens up or down. (Lesson 9-3)

22.  $y = -\frac{1}{3}x^2 - 6x + 1$

23.  $y = 0.5x - 2x^2$

24. **Skill Sequence** In Parts a–d, simplify each statement. (Lesson 2-2)

a.  $\frac{-4+x}{2a} + \frac{-4-x}{2a}$

b.  $\frac{-b+y}{2a} + \frac{-b-y}{2a}$

c.  $\frac{-b+\sqrt{z}}{2a} + \frac{-b-\sqrt{z}}{2a}$

d.  $\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$

- e. What does Part d tell you about the solutions to a quadratic equation?

### EXPLORATION

25. Create a parabola in the following way. Take a plain sheet of notebook paper and draw a dark dot in the middle of the paper. Draw a line anywhere on the paper that is parallel to the bottom of the paper. Make sure the line stretches to both edges of the paper. Now fold the paper so that the dot falls on the line. Unfold the paper, and then fold the paper so that the dot falls on another place on the line. Repeat this 20 times so that the dot has fallen in different places on the line each time. The folds should outline a parabola that can be seen by unfolding the paper. Take another sheet and see what happens if the dot is farther from the line or closer to the line than the dot you used the first time.



Softball was invented as an indoor sport by George Hamock of the Chicago Board of Trade in 1887.

Source: The History Channel

QY ANSWER

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