Lesson

9-4 Quadratics and Projectiles

Vocabulary

force of gravity
initial upward velocity
initial height

BIG IDEA Assuming constant gravity, both the path of a projectile and the height of a projectile over time can be described by an equation of the form $y = ax^2 + bx + c, a \neq 0$.

A *projectile* is an object that is thrown, dropped, or launched, and then proceeds with no additional force on its own. A ball thrown up into the air is considered a projectile.

Mental Math

Estimate to the nearest 10.

a. 24% of 82

b. 5% of 206

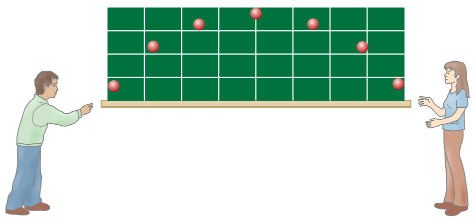
c. 61% of 92

Equations for the Paths of Projectiles

When there is a constant force of gravity, the path of a projectile is a parabola. The parabola shows the height of the projectile as a function of the *horizontal distance* from the launch.

Activity

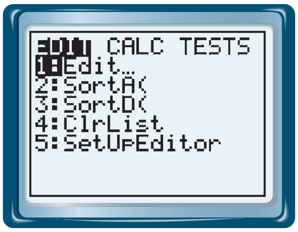
Step 1 A classroom board should be partitioned into rectangles by drawing evenly spaced lines, as shown. Work in a group and assign two students the tasks of tossing and catching a ball. Position the tosser at the left end of the board and the catcher at the right end of the board. During the experiment, the tosser will toss the ball to the catcher so that it does not go higher than the top of the board. For each vertical division line on the board, assign a student to act as a spotter. The diagram at the right would require 7 spotters. When the ball is tossed, these spotters will observe the ball's height when it crosses their vertical line.

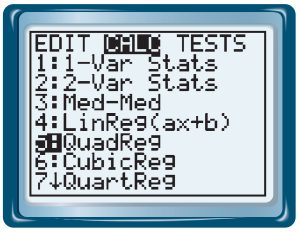


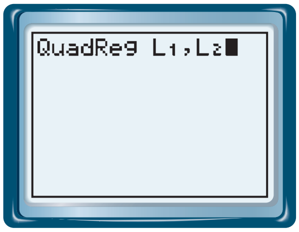
Step 2 Toss the ball while the spotters note its height as it passes each of the vertical lines. (It may take a few tosses to get everything right.) Following the toss, each spotter should mark the approximate point on the board at which the ball passed his or her vertical line. Using the board as a coordinate system, measure the horizontal and vertical distances to each point. Record the results.

x = horizontal distance ball has traveled	?	?	?	?	?	?	?	?	?
y = height of the ball	?	?	?	?	?	?	?	?	?

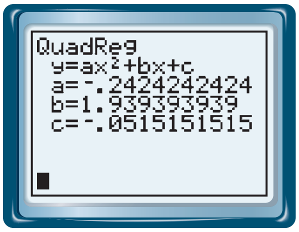
Step 3 Enter the data into lists L1 and L2 on your calculator. Perform *quadratic regression* on the data. Quadratic regression fits a parabola "of best fit" to three or more points in the plane. Here is how to apply quadratic regression on one calculator.







The calculator returns coefficients *a*, *b*, and *c* of the quadratic equation that most closely fit the data entered into lists L1 and L2.



Step 4 Next, set up a reasonable plot window to fit your data. Enter the equation obtained in Step 3 into your Y= menu. Plot the scatterplot and function on the same grid. How close is the scatterplot to the parabola of best fit?

Equations for the Heights of Projectiles over Time

The graph of Galileo's formula $d = -16t^2$ is also a parabola. That parabola describes the height of the projectile as a function of the *length of time* since the projectile was launched.

When a projectile is launched, several factors determine its height above the ground at various times:

1. The **force of gravity** pulls the projectile back to Earth. By Galileo's formula, gravity pulls the projectile towards Earth $16t^2$ feet in t seconds.
2. The **initial upward velocity** with which the projectile is thrown or shot contributes to its height at time t. We use v to stand for the initial velocity. In t seconds, the projectile would go vt feet if there were no gravity.
3. The **initial height** of the projectile. We call this height s (for starting height).

Adding all these forces together yields a formula for the height of a projectile over time. Notice that the force of gravity is downward, so it has a negative effect on the height, while the launch velocity is considered to be upward and positive.

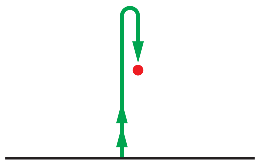
General Formula for the Height of a Projectile over Time

Let h be the height (in feet) of a projectile launched with an initial upward velocity v feet per second and an initial height of s feet. Then, after t seconds, $h = -16t^2 + vt + s$.

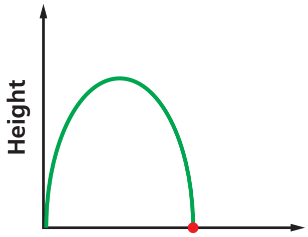
Since 16 feet ≈ 4.9 meters, if the units are in meters in the formula above, then $h = -4.9t^2 + vt + s$.

It is very easy to confuse the graph showing the height of a projectile as a function of time with a graph that represents the object's path, because they both are parabolas. However, even when a ball is tossed straight up and allowed to fall, its graph of height as a function of time is a parabola.

Ball's Path



Ball's Height Graphed as a Function of Time



Example 1

A ball is launched from an initial height of 6 feet with an initial upward velocity of 32 feet per second.

a. Write an equation describing the height h in feet of the ball after t seconds.

b. How high will the ball be 2 seconds after it is thrown?

c. What is the maximum height of the ball?

Solutions

a. Since units are provided in feet, use the general formula $h = -16t^2 + vt + s$. Substitute 6 for s and 32 for v.

$$h = -16t^2 + 32t + 6$$

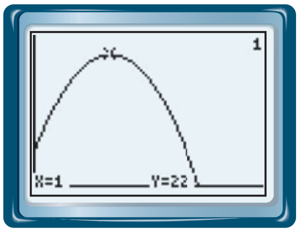
b. Substitute 2 for t in $h = -16t^2 + 32t + 6$.

$$h = -16(2)^2 + 32(2) + 6$$
$$= -16(4) + 64 + 6$$
$$= -64 + 64 + 6$$
$$= 6$$

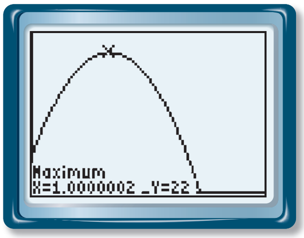
In 2 seconds, the ball will be 6 feet high.

c. Use a graphing calculator. Plot $y = -16x^2 + 32x + 6$. A graph of this function using the window $0 \leq x \leq 3$, and $0 \leq y \leq 25$ is shown at the right. The maximum height is the greatest value of h shown on the graph, the y-coordinate of the vertex. Trace along the graph, and read the y-coordinate as you go. When we did this using the window $0 \leq x \leq 3$, and $0 \leq y \leq 25$, our trace showed that (0.989, 21.998) and (1.021, 21.993) are on the graph. So try $t = 1$. This gives $h = 22$.

The maximum height reached is 22 feet.



Check You can verify the maximum height by using the MAXIMUM or VERTEX command on your calculator. The screen at the right shows the vertex as (1, 22).



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Example 2

An object is dropped from an initial height of 90 meters.

a. Write a formula describing the height of the object (in meters) after *t* seconds.

b. After how many seconds does the object hit the ground?

c. What is the maximum height of the object?

Solutions

a. Because units are provided in meters, use the general formula $h = -4.9t^2 + v \cdot t + s$.

Substitute 90 for s and 0 for v.

$$h = -4.9t^2 + \underline{\ ?\ } \cdot t + \underline{\ ?\ }$$
$$= -4.9t^2 + \underline{\ ?\ }$$

b. The value(s) of t corresponding to $h = 0$ must be found when the object hits the ground.

$$0 = -4.9t^2 + \underline{\ ?\ }.$$
$$4.9t^2 = \underline{\ ?\ }$$
$$t^2 \approx \underline{\ ?\ }$$

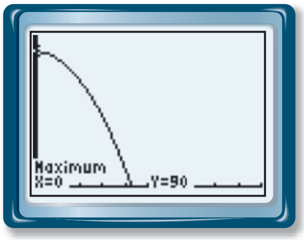
Because t is positive (it measures time after launch), ignore the negative square root.

$$t \approx \underline{\ ?\ }$$

(*continued on next page*)

You should find that the object hits the ground in about 4.3 seconds.

c. A graph of $y = -4.9x^2 + 90$ is shown at the right. The graph shows that the object is farthest from the ground when it is launched. That is when $t = 0$. Then $h = -4.9t^2 + 90 = -4.9(0)^2 + 90 = 90$.



The maximum height is 90 meters. We knew this because the object was dropped from this height, and the equation and graph confirm it.

If a projectile is thrown upward and comes back down to Earth, then it will reach some heights twice.

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Example 3

Suppose a ball is thrown upward with an initial velocity of 22 meters per second from an initial height of 2 meters.

a. Write a formula for the height in meters of the ball after *t* seconds.

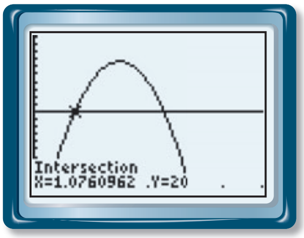
b. Estimate when the ball is 20 meters high.

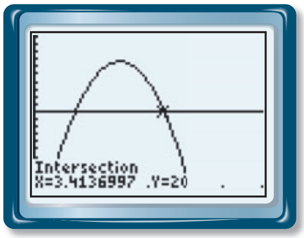
Solutions

a. Because units are provided in meters, use $h = -4.9t^2 + vt + s$. Substitute 2 for *s* and 22 for *v*.

h = __?__

b. The values of *t* corresponding to $h = 20$ must be found. Graph the equation you found in Part a on the window $0 \le x \le 6$, $10 \le y \le 30$. Draw a horizontal line $y = 20$ to indicate $h = 20$ feet. Use the INTERSECT command on your calculator to find both intersections. Our calculator shows that when $x \approx$ __?__ and $x \approx$ __?__, $y \approx 20$.





The ball is 20 meters off the ground at about __?__ seconds and __?__ seconds after being thrown.

A method for finding exact answers without finding the intersections of two graphs is discussed in Lesson 9-5.

Questions

COVERING THE IDEAS

1. In your own words, define the term projectile. Give several real-life examples of projectiles.

2. Use quadratic regression on your calculator to find an equation for the parabola passing through the points $(0, 6)$, $(5, 30)$, and $(10, 6)$.

3. A ball is thrown from an initial height of 5 feet with an initial upward velocity of 30 feet per second.
   a. Write a function describing the height of the ball after t seconds.
   b. How high will the ball be 2 seconds after it is thrown?
   c. What is the maximum height of the ball?

4. Suppose a ball is batted with an initial upward velocity of 26 meters per second from an initial height of 1 meter.
   a. Write a function describing the height in meters of the ball after t seconds.
   b. Estimate when the ball is 5 meters high.

Los Angeles's Vladimir Guerrero hits a solo home run in the fourth inning of a baseball game in Oakland, California.

Source: Associated Press

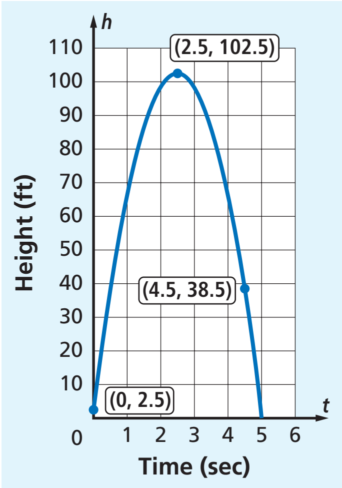
5. An object is dropped from an initial height of 40 feet.
   a. What is the object's initial velocity? What is its maximum height?
   b. Write a formula for the height h in feet of the object after t seconds.
   c. After how many seconds does the object hit the ground?

6. An object is dropped from an initial height of 150 meters.
   a. Write a formula for the height h in meters of the object after t seconds.
   b. After how many seconds does the object hit the ground?
   c. What is the maximum height of the object?

APPLYING THE MATHEMATICS

7. Use quadratic regression on your calculator to answer this question. A football kicker attempts to kick a 40-yard field goal. The kicker kicks the football from a height of 0 feet above the ground. The football is 26 feet above the ground at its peak (the vertex) at a distance of 22 yards from the kicker. The height of the crossbar (the bottom bar of the goal post) is 10 feet off the ground. Assuming the ball is kicked straight, will the kick clear the crossbar of the goal post? (*Hint:* Use the information given to determine a third point on the parabola and use quadratic regression.)

8. Refer to the graph at the right. It shows the height h in feet of a soccer ball t seconds after it is drop-kicked into the air.



   a. What is the approximate height of the soccer ball at 1 second? At 4 seconds?
   b. What is the greatest height the ball reaches?
   c. At what times is the ball 38.5 feet high?
   d. Approximately how long is the ball in the air?
   e. For how many seconds is the ball more than 38.5 feet above the ground?
   f. What does the h-intercept represent in this situation?
   g. What does the t-intercept represent in this situation?

There were approximately 321,555 high school girl soccer players in the 2005–2006 school year.

Source: National Federation of High School Associations

9. A small rocket is shot from the edge of a cliff. Suppose that after x seconds, the rocket is y meters above the cliff, where $y = 25x - 5x^2$.
   a. Graph this equation using the window $0 \leq x \leq 8$, $-5 \leq y \leq 40$.
   b. What is the greatest height the rocket reaches?
   c. How far above the edge of the cliff is the rocket after 4 seconds?
   d. Between which times is the rocket more than 20 feet above the cliff's edge?
   e. What is the height of the rocket after 6 seconds?
   f. At approximately what time does the rocket fall below the height of the cliff's edge?

10. A ball is thrown from an initial height of 7 feet. After 5 seconds in the air, the ball reaches a maximum height of 18 feet above the ground.
   a. What third point on the graph can be deduced from this information?
   b. Use quadratic regression to find a formula for the height in feet of the ball after t seconds.
   c. Use the formula to find the initial velocity of the ball.
   d. Use the graph to approximate how long the ball is in the air.

REVIEW

In 11 and 12, an equation is given.

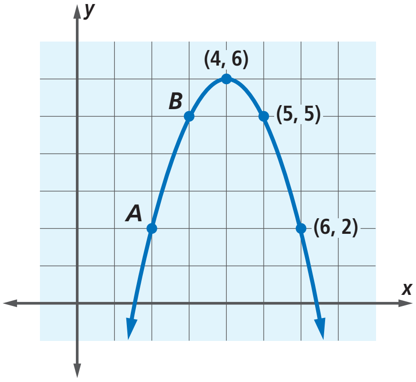
a. Make a table of values for integer values of x from −3 to 3.

b. Graph the equation. (Lessons 9-3, 9-2)

11. $y = \frac{1}{4}x^2$

12. $y = 8x - 3x^2$

In 13 and 14, use the graph of the parabola at the right. (Lessons 9-3, 9-1)



13. Use the symmetry of the parabola to find the coordinates of points A and B.

14. Write an equation for the axis of symmetry.

15. **Multiple Choice** Which of the following is *not* equal to uv^2? (Lessons 8-5, 8-2)

A $u \cdot v \cdot v$ **B** $(u)v^2$ **C** $u(v)^2$ **D** $(uv)^2$

16. The table below estimates the number of calories people of various weights burn per minute while participating in various activities. (Lessons 6-8, 6-5, 3-4)

	Weight (lb)			
Activity	**105–115**	**127–137**	**160–170**	**180–200**
Full-court Basketball	9.8	11.2	13.2	14.5
Jogging (5 mph)	8.6	9.2	11.5	12.7
Running (8 mph)	10.4	11.9	14.2	17.3
Volleyball	7.8	8.9	10.5	11.6
Bicycling (10 mph)	5.5	6.3	7.8	14.5

Source: www.coolnurse.com

The state of California has produced the most high school All-American boy's basketball players (72) through the 2005–2006 season.

Source: mcdepk.com

a. Suppose Vince, who weighs 168 pounds, works out by jogging and then playing basketball. Let x = the number of minutes he jogs, and let y = the number of minutes he plays basketball. If he burns a total of 445 calories, write an equation in standard form that describes x and y.

b. Find the x- and y-intercepts of the line from Part a.

c. If Vince plays basketball for 25 minutes, use your equation to calculate how long he must jog to burn 445 calories.

EXPLORATION

17. Infinitely many parabolas have x-intercepts at 0 and 6. Find equations for three such parabolas.