Chapter 4 Summary and Vocabulary

- A **matrix** is a rectangular array of objects. Matrices are frequently used to store data and to represent **transformations**. Matrices can be added or subtracted if they have the same **dimensions**. Addition of matrices can be used to obtain translation images of figures.

- Any matrix can be multiplied by a real number, called a *scalar*. Multiplying each element in the matrix by the scalar yields the **scalar product**. However, not all matrices can be multiplied by other matrices. The product of two matrices exists only if the number of columns of the left matrix equals the number of rows of the right matrix. The element in row r and column c of AB is the product of row r of A and column c of B. Matrix multiplication is associative but not commutative.

- Matrices with 2 rows can represent points, segments, lines, polygons, and other figures in the coordinate plane. Multiplying such a matrix by a 2×2 matrix on its left may yield a transformation image of the figure. Transformations for which 2×2 matrices are given in this chapter include **reflections**, **rotations**, **size changes**, and **scale changes**. They are summarized on the next two pages. The rotation of 90° about the origin is a particularly important transformation. Based on that transformation, it can be proved that two nonvertical lines are perpendicular if and only if the product of their slopes is –1.

- The set of 2×2 matrices is closed under multiplication. The **identity matrix** for multiplying 2×2 matrices is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The **identity transformation** maps any figure onto itself.

  The **Matrix Basis Theorem** provides a way to generate and remember matrices for transformations. When a transformation A is represented by a 2×2 matrix, if $A(1, 0) = (x_1, y_1)$ and $A(0, 1) = (x_2, y_2)$, then A has the matrix $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$.

Vocabulary

4-1
*matrix
element
*dimensions
equal matrices
point matrix

4-2
*matrix addition, sum of two matrices
scalar multiplication, scalar product
difference of two matrices

4-3
row-by-column multiplication
*matrix multiplication
matrix product

4-4
*transformation
preimage
image
*size change
center of a size change
magnitude of a size change
*identity matrix
*identity transformation
similar

4-5
*scale change
horizontal magnitude
vertical magnitude
stretch
shrink

4-6
reflection image of a point over a line
reflecting line, line of reflection
*reflection

Vocabulary

4-7
*composite of two transformations

4-8
*rotation
center of a rotation
magnitude of a rotation

4-10
*translation

- **Reflections, rotations,** and **translations** preserve distance. A size change S_k multiplies distances by k. These properties can be proved using the Pythagorean Distance Formula for the distance d between two points (x_1, y_1) and (x_2, y_2):

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}.$$

Matrices for many specific transformations were discussed in this chapter.

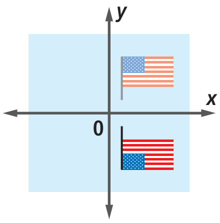
- **Transformations Yielding Images Congruent to Preimages**

Reflections:

over x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

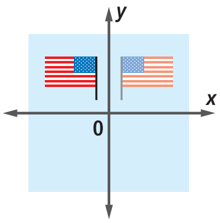
$r_x: (x, y) \rightarrow (x, -y)$



over y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

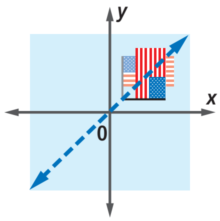
$r_y: (x, y) \rightarrow (-x, y)$



over the line $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

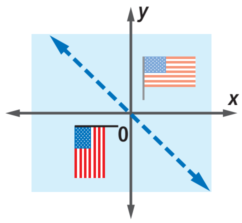
$r_{y=x}: (x, y) \rightarrow (y, x)$



over the line $y = -x$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$r_{y=-x}: (x, y) \rightarrow (-y, -x)$

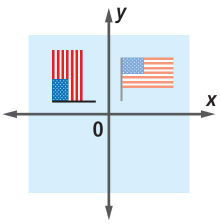


Rotations with center (0, 0):

magnitude 90°

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

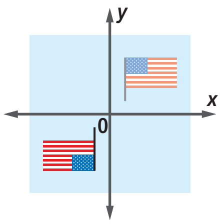
$R_{90}: (x, y) \rightarrow (-y, x)$



magnitude 180°

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

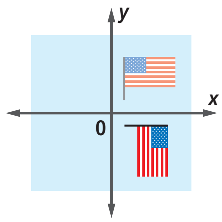
$R_{180}: (x, y) \rightarrow (-x, -y)$



magnitude 270°

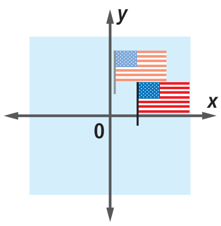
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$R_{270}: (x, y) \rightarrow (y, -x)$



Translations:

No general matrix
$T_{h,k}: (x, y) \rightarrow (x + h, y + k)$

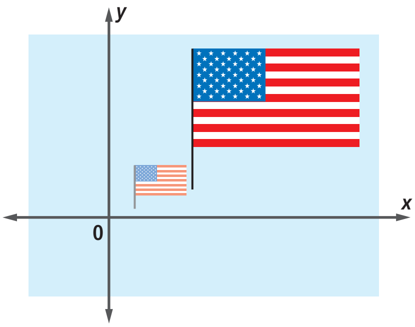


Transformations Yielding Images Similar to Preimages

Size changes with center (0, 0) and magnitude k:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$S_k: (x, y) \rightarrow (kx, ky)$$

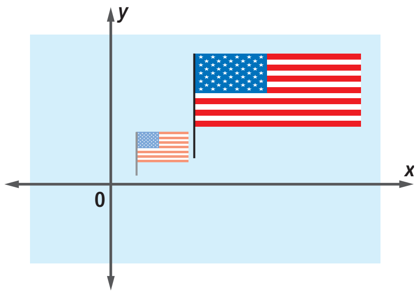


Other Transformations

Scale changes with horizontal magnitude a and vertical magnitude b:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$S_{a,b}: (x, y) \rightarrow (ax, by)$$



Theorems

Size Change Theorem (p. 244)
Pythagorean Distance Formula (p. 245)
Scale Change Theorem (p. 251)
Matrix for r_y Theorem (p. 256)
Matrix Basis Theorem (p.257)
Matrices for r_x, $r_{y=x}$, and $r_{y=-x}$ Theorem (p. 259)
Matrices and Composites Theorem (p. 265)
Composite of Rotations Theorem (p. 270)
Matrix for R_{90} Theorem (p. 270)
Perpendicular Lines and Slopes Theorem (pp. 274, 276)
Parallel Lines and Translations Theorem (p. 282)

Chapter 4 Self-Test

Take this test as you would take a test in class. You will need a calculator. Then use the Selected Answers section in the back of the book to check your work.

1. Write a matrix to represent polygon *HULK* if $H = (-2, 4)$, $U = (5, 1)$, $L = (-2, -2)$, and $K = (-4, 2)$.

2. One day on a Veggie-Air flight from Iceburg, there were 16 first-class and 107 economy passengers going to Jicamaport, 4 first-class and 180 economy passengers bound for Okraville, and 2 first-class and 321 economy passengers flying to Potatotown.

a. Write a 2×3 matrix to store this information. Include appropriate column and row headings.

b. Write a 3×2 matrix to store this information. Include appropriate column and row headings.

In 3–6, use matrices *A*, *B*, and *C* below.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -2 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad B = \begin{bmatrix} 5 & -2 & -1 \\ \frac{1}{8} & 6 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 17 & 4 \\ \frac{1}{3} & \sqrt{2} \end{bmatrix}$$

3. Determine which of the following products exist: *AB*, *BA*, *AC*, *CA*, *BC*, and *CB*.

4. If possible, find *BA*. If it is not possible, explain why.

5. If possible, find $A - C$. If it is not possible, explain why.

6. Calculate $\frac{1}{3}B$.

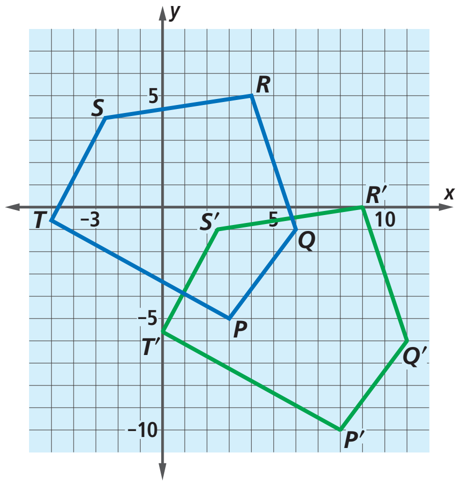
7. Why is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ called the identity matrix?

8. Are the two lines with equations $y = 5x - 3$ and $y = \frac{1}{5}x + 2$ perpendicular? Explain your answer.

9. Find an equation for the line through $\left(\frac{1}{4}, -1\right)$ that is perpendicular to $y = \frac{1}{7}x + 4$.

10. Calculate the matrix for $r_y \circ R_{270}$.

In 11–12, refer to the graph below.



11. What translation maps *PQRST* onto *P′Q′R′S′T′*?

12. Graph the image of *PQRST* under the transformation r_x.

13. The 4-Star Movie Theater has four screens. The number and type of theater attendees is summarized in the following matrix for one show time of four different movies.

	Children	Adults	Students
Movie 1	38	135	169
Movie 2	84	101	152
Movie 3	84	118	135
Movie 4	67	236	34

Ticket prices for children, adults, and students are \$4.00, \$6.50, and \$5.00, respectively. Use matrix multiplication to determine the total ticket sales revenue for each movie.

14. Savannah Reed and Denise Wright have decided to merge their book inventories and hold a book sale. If the matrices below represent each person's inventory, write a matrix for the inventory of books after merging.

	Savannah Fiction	Savannah Nonfiction	Denise Fiction	Denise Nonfiction
Paperbacks	42	15	5	2
Hardbacks	10	2	3	1
Audiotapes	7	1	0	3
Audio CDs	4	0	2	0

15. Show that multiplying $A = \begin{bmatrix} 2 & -9 \\ 7 & 5 \end{bmatrix}$ by the scalar 2 is equivalent to multiplying A on the left by the matrix for S_2.

16. Write a matrix you can use to apply the transformation T, where $T: (x, y) \rightarrow (-y, x)$, to a figure in the coordinate plane.

In 17 and 18, $\triangle XYZ$ has vertices $X = (-4, 5)$, $Y = (2, 6)$, and $Z = (-3, 1)$.

17. Write a sentence that describes this matrix multiplication geometrically:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 2 & -3 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 6 & 1 \end{bmatrix}.$$

18. Graph $\triangle XYZ$ and $R_{270}(\triangle XYZ)$.

19. Find the matrix of the image of $\begin{bmatrix} 2.5 & 1 & -6 \\ -5 & 4 & -12 \end{bmatrix}$ under each transformation.

a. $S_{4,3}$ **b.** $T_{2.5,-1}$

True or False In 20 and 21, if the statement is false, give an example to show that it is false.

20. A line and its translation image are always parallel.

21. A line and its rotation image are always perpendicular.

22. Consider the transformation S_7.

a. What is the image of (a, b) under S_7?

b. If $P = (2, 5)$ and $Q = (3, 9)$, show that the distance between $S_7(P)$ and $S_7(Q)$ is 7 times the distance between P and Q.

23. Find an equation for the perpendicular bisector of $\overline{PQ}$ in Question 22.

Chapter 4 Chapter Review

SKILLS
PROPERTIES
USES
REPRESENTATIONS

SKILLS Procedures used to get answers

OBJECTIVE A Write matrices for points and polygons. (Lesson 4-1)

1. Write a matrix to represent each point.
   a. $U = (4, 5)$
   b. $E = (3, -4)$
   c. $D = (-4, 1)$
   d. $G = (-2, -4)$
   e. $F = (3, 1)$

2. Write a matrix to represent polygon $FUDGE$, if its vertices are those defined in Question 1.

OBJECTIVE B Add, subtract, and find scalar multiples of matrices. (Lesson 4-2)

In 3–6, let $A = \begin{bmatrix} 2 & 2 & 5 \\ 6 & 4 & -2 \\ 0 & -3 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 & 7 \\ 3 & 3 & -5 \\ 4 & 1 & -10 \end{bmatrix}$. Calculate.

3. $A - B$
4. $A - 3B$
5. $B - 2A$
6. $B + A$

In 7 and 8, find p and q.

7. $2\begin{bmatrix} 4 & p \\ q & -3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5q & 7 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 21 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 8 & -2 \\ -3 & p \end{bmatrix} - 2\begin{bmatrix} q & 4 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 11 & 7 \end{bmatrix}$

OBJECTIVE C Multiply matrices. (Lesson 4-3)

In 9–12, multiply the matrices if possible.

9. $\begin{bmatrix} 4 & 1 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 2 & 5 \\ 6 & 4 & -2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & 3 \\ 4 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 3 & -1 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

In 13 and 14, find p and q.

13. $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 16 \\ -4 \end{bmatrix}$

14. $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$

OBJECTIVE D Determine equations of lines perpendicular to given lines. (Lesson 4-9)

15. Find an equation of the line through $(2, -5)$ perpendicular to the line $y = \frac{1}{3}x + 1$.

16. Find an equation of the line through $(4, -6)$ perpendicular to the line $y = 5$.

17. Given $A = (4, 7)$ and $B = (-6, 1)$, find an equation for the perpendicular bisector of $\overline{AB}$.

18. Consider two lines. One is the image of the other under R_{90}. The slope of one of the lines is $\frac{1}{8}$. What is the slope of the other line?

PROPERTIES Principles behind the mathematics

OBJECTIVE E Recognize properties of matrix operations. (Lessons 4-2, 4-3, 4-7)

In 19–22, a statement is given.

a. Is the statement true or false?

b. Give an example to support your answer.

19. Matrix multiplication is associative.

20. Matrix multiplication is commutative.

21. Matrix subtraction is commutative.

22. Scalar multiplication of matrices is commutative.

In 23 and 24, suppose *Y* and *P* are matrices. *Y* has dimensions 1×7 and *P* has dimensions $m \times n$.

23. If the sum $Y + P$ exists, what are the values of m and n?

24. If the product PY exists, what is the value of n?

25. What matrix is the identity for multiplication of 2×2 matrices?

OBJECTIVE F Recognize relationships between figures and their transformation images. (Lessons 4-4, 4-5, 4-6, 4-8, 4-9, 4-10)

In 26 and 27, fill in the blank with A, B, or C to make a true statement.

A not necessarily similar or congruent

B similar, but not necessarily congruent

C congruent

26. A figure and its size change image are _?_.

27. A figure and its reflection image are _?_.

28. Give an example to show that a figure and its image under $S_{3,\frac{2}{3}}$ are not similar.

29. Use the Pythagorean Distance Formula to show that S_2 multiplies distances by 2.

30. Find an equation for the image of the line $2x + 3y = 60$ under R_{90}.

31. Repeat Question 30 if the transformation is the translation $T_{-1,3}$.

OBJECTIVE G Relate transformations to matrices, and vice versa. (Lessons 4-4, 4-5, 4-6, 4-7, 4-8, 4-10)

32. Translate the matrix equation $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$ into English by filling in the blanks.

The image of the point __?__ under a rotation with center __?__ and magnitude _?_ is the point __?__.

33. Multiply the matrix for $r_{y=x}$ by itself, and tell what transformation the product represents.

34. Write a matrix for a scale change with horizontal magnitude 5 and vertical magnitude 3.5.

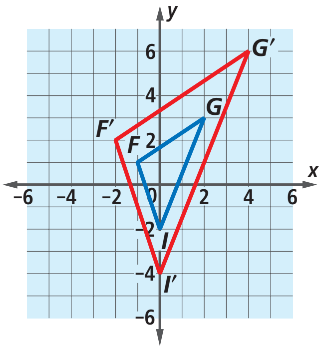
35. a. Calculate a matrix for $R_{180} \circ r_x$.

b. What single transformation corresponds to your answer?

36. a. Find two reflections whose composite is R_{180}.

b. Use matrix multiplication and your answer to Part a to generate the matrix for R_{180}.

37. a. What size change maps FIG onto $F'I'G'$ as shown below?



b. Explain how to use a matrix operation to transform FIG to $F'I'G'$.

Multiple Choice In 38–40, choose the matrix for each transformation.

A $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$

D $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **F** $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

38. $r_{y=x}$ **39.** S_3 **40.** R_{90}

41. Find the image of $\begin{bmatrix} 2 & -1 & 4 & 3 \\ 0.7 & 0 & -1 & 5 \end{bmatrix}$ under r_x.

42. $HARP$ has coordinates $H = (-2, 2)$, $A = (-1, -2)$, $R = (0, 0)$, and $P = (1, 4)$. Find the matrix of $HARP$ under R_{270}.

43. Find the matrix of $\begin{bmatrix} 0 & -8 & 6 \\ 4 & 0 & 4 \end{bmatrix}$ under $S_{0.5}$.

OBJECTIVE H **Given their slopes, determine whether lines are parallel or perpendicular to each other, and vice versa.** (Lessons 4-9, 4-10)

44. Line ℓ has equation $y = 3x$, and line m has equation $y = kx$. If $\ell \perp m$, find the value of k.

45. Suppose $A = (0, 3)$, $B = (2, 4)$, $C = (7, 8)$, and $D = (2, 9)$. Is $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$? Explain.

46. Multiple Choice Line a has slope –4. Which of the following is the slope of a line perpendicular to a?

A 4 **B** $\frac{1}{4}$ **C** $-\frac{1}{4}$ **D** –4

47. Lines j and k are parallel. j has slope 8 and k passes through the point (0, 0). Find another point on k.

48. Let $\triangle NSA$ be represented by the matrix $\begin{bmatrix} 152 & -12 & 87 \\ 16 & 113 & -23 \end{bmatrix}$. Let $\triangle N'S'A' = R_{270}(\triangle NSA)$.

a. What is the slope of $\overleftrightarrow{NA}$?

b. What is the product of the slopes of $\overleftrightarrow{NA}$ and $\overleftrightarrow{N'A'}$?

c. Use your answers to Parts a and b to find the slope of $\overleftrightarrow{N'A'}$.

USES Applications of mathematics in real-world situations

OBJECTIVE I **Use matrices to store data.** (Lesson 4-1)

49. In 2005, the average loss for people who were victims of three common online scams was \$240 for credit or debit card fraud, \$410 for nondelivery of merchandise, and \$2000 for investment fraud. In 2006, the amounts lost to these scams averaged \$427.50, \$585, and \$2694.99, respectively. Store these data in a 2 × 3 matrix.

50. The recommended daily allowance (RDA) of vitamin K is 60 µg for a 9- to 13-year-old male, 75 µg for a 14- to 18-year-old male, and 120 µg for a 19- to 30-year-old male. (Note: µg means micrograms, or one-millionth of a gram.) The RDA of thiamin is 0.9 mg for a 9- to 13-year-old male, 1.2 mg for a 14- to 18-year-old male, and 1.2 mg for a 19- to 30-year-old male. The RDAs of vitamin C for these age categories are 45 mg, 75 mg, and 90 mg, respectively. The RDAs of niacin for these age categories are 12 mg, 16 mg, and 16 mg, respectively. Store these data in a 3 × 4 matrix.

In 51 and 52, the matrix gives the time (in minutes and seconds) behind Lance Armstrong's time each competitor in the Tour de France finished in each of three years.

	2003	2004	2005
Lance Armstrong	0:00	0:00	0:00
Jan Ulrich	1:01	8:50	6:21
Francisco Mancebo	19:15	18:01	9:59

51. Which element represents the time behind Armstrong's time that Jan Ullrich finished in 2003?

52. How much time behind Jan Ullrich's time did Francisco Mancebo finish in 2005?

OBJECTIVE J **Use matrix addition, matrix multiplication, and scalar multiplication to solve real-world problems.** (Lessons 4-2, 4-3)

53. The matrices below contain box office data for three popular movie series. One matrix contains the movies' domestic gross earnings, in millions of dollars, for both the first and second movie in the series, while the other matrix contains the movies' foreign gross earnings.

Domestic Gross (10^6 dollars)

	Spiderman	*The Matrix*	*Harry Potter*
Original	404	171	318
Sequel	374	282	262

Foreign Gross (10^6 dollars)

	Spiderman	*The Matrix*	*Harry Potter*
Original	418	285	659
Sequel	410	457	617

a. Calculate the matrix that stores the worldwide total amount of money each movie made (in millions of dollars).

b. Of these movies, which made the most money worldwide?

c. Which sequel made more money worldwide than its original movie?

54. Suppose the New York Yankees and Seattle Mariners both decided to raise their ticket prices by 5%. Prices of tickets in dollars in 2008 for three tiers are given in the matrix below.

	Bleachers	Premium Box	Upper Deck
Yankees	14	95	80
Mariners	7	60	20

a. What scalar multiplication will yield the new ticket prices?

b. Find a matrix that stores the new ticket prices for each team.

55. In basketball, a free throw is worth 1 point, a shot made from inside the three-point arc is worth 2 points, and a shot made from behind the three-point arc is worth 3 points. Suppose that in one game, Brenda made 9 free throws, 11 shots from inside the three-point arc, and 2 shots from behind the three-point arc. In the same game, Marisa made 5 free throws, 7 shots from inside the three-point arc, and 5 shots from behind the three-point arc. Write a matrix B for the number of each type of basket each player made and a matrix P for the number of points the baskets are worth, then calculate BP to find the total points each player scored.

56. An office supply store sells three different models of graphing calculators. Model X sells for \$150, model Y sells for \$130, and model Z sells for \$100. The following matrix multiplication represents a teacher's order at the store.

$$\begin{bmatrix} 5 & 11 & 7 \end{bmatrix} \begin{bmatrix} 150 \\ 130 \\ 100 \end{bmatrix}$$

a. How many of each model did the teacher order?

b. What is the total cost of the order?

REPRESENTATIONS Pictures, graphs, or objects that illustrate concepts

OBJECTIVE K Graph figures and their transformation images. (Lessons 4-4, 4-5, 4-6, 4-7, 4-8, 4-10)

57. a. Graph the polygon *HELP* described by the matrix $\begin{bmatrix} -3 & 1 & 1 & -3 \\ 2 & 1 & 5 & 7 \end{bmatrix}$.

b. Use matrix multiplication to find the image of *HELP* under $S_{2,3}$.

c. Graph the image of *HELP*.

d. Is the image similar to the preimage? Explain why or why not.

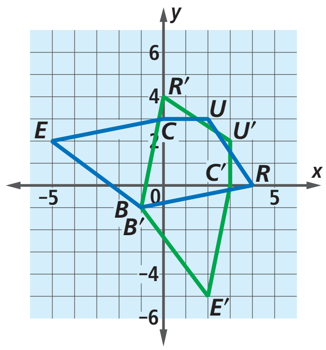
58. A popular road atlas company uses a 1:7,500,000 scale to represent roads on a map. Consider the actual road to be the preimage. Write a matrix that could be used to transform the road to its map representation.

59. Consider $\triangle ABC$ defined by $\begin{bmatrix} -5 & 0 & -3 \\ 1 & 0 & -3 \end{bmatrix}$.

a. Graph $\triangle ABC$ and $\triangle A'B'C'$, the image of $\triangle ABC$ under $r_{y=x} \circ r_x$.

b. What single transformation maps $\triangle ABC$ to $\triangle A'B'C'$?

In 60 and 61, refer to polygon BRUCE and its image *B′R′U′C′E′* shown below.



60. a. What transformation maps *BRUCE* onto *B′R′U′C′E′*?

b. Show how the coordinates of *B′R′U′C′E′* can be derived by matrix multiplication.

61. a. Write the image *B″R″U″C″E″* of *B′R′U′C′E′* under r_x as a matrix and draw the image.

b. Is *B″R″U″C″E″* congruent to *B′R′U′C′E′*? Explain.

c. Are the two images similar? Explain.