Lesson

4-10 Translations and Parallel Lines

Vocabulary

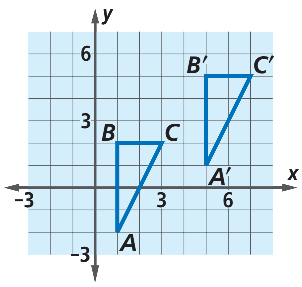
translation

BIG IDEA By adding matrices, translation images of figures in the plane can be described.

In this chapter you have found images for many transformations by multiplying 2 × 2 matrices. There is one transformation for which images can be found by *adding* matrices.

Translations

Consider $\triangle ABC$ and its image $\triangle A'B'C'$ at the left below. Matrices M and M' representing the vertices of these triangles are given at the right below.



$\triangle ABC$

$$M = \begin{bmatrix} 1 & 1 & 3 \\ -2 & 2 & 2 \end{bmatrix}$$

$\triangle A'B'C'$

$$M' = \begin{bmatrix} 5 & 5 & 7 \\ 1 & 5 & 5 \end{bmatrix}$$

Activity

Use the matrices M and M' above.

Step 1 Calculate $D = M' - M$.

Step 2 What does the first row of D represent?

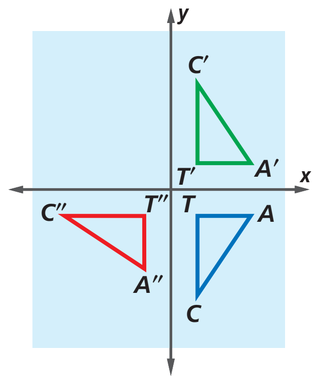
Step 3 What does the second row of D represent?

Step 4 Calculate $D + M$. What does $D + M$ represent?

Step 5 Complete this sentence: If (x, y) is any point in the preimage, then its image is ___?___.

The transformation in the Activity is an example of a *translation* or *slide*.

Mental Math



Use the graph above to identify the transformation that maps

a. $\triangle CAT$ onto $\triangle C'A'T'$.

b. $\triangle C'A'T'$ onto $\triangle C''A''T''$.

c. $\triangle CAT$ onto $\triangle C''A''T''$.

Definition of Translation

The transformation that maps (x, y) onto $(x + h, y + k)$ is a **translation** of h units horizontally and k units vertically, and is denoted by $T_{h,k}$.

Using mapping notation, $T_{h,k}: (x, y) \rightarrow (x + h, y + k)$.

Using $f(x)$ notation, $T_{h,k}(x, y) = (x + h, y + k)$.

In the figure on the previous page, $\triangle A'B'C'$ is the image of $\triangle ABC$ under the translation $T_{4,3}$.

Reading Math

The Latin prefix *trans* means "across." When we translate a geometric figure, we move it across the plane. Another term in this chapter with the same prefix is *transformation*.

Matrices for Translations

There is no single matrix representing a specific translation because the dimensions of the translation matrix depend on the figure being translated. Example 1 shows you how to find image coordinates with and without matrices.

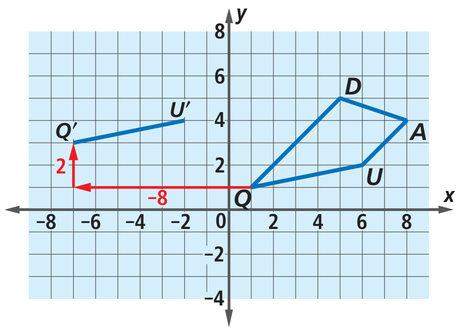
GUIDED

Example 1

A quadrilateral has vertices $Q = (1, 1)$, $U = (6, 2)$, $A = (8, 4)$, and $D = (5, 5)$.

a. Find its image under the translation $T_{-8,2}$.

b. Graph the image and preimage on the same set of axes.



Solution

a. Using $f(x)$ notation:

$T_{-8,2}(x, y) = (x - 8, y + 2)$

$Q' = T_{-8,2}(1, 1) = (1 - 8, 1 + 2) = (-7, 3)$

$U' = T_{-8,2}(6, 2) = (\underline{?}, \underline{?}) = (-2, 4)$

$A' = T_{-8,2}(8, 4) = (\underline{?}, \underline{?}) = (\underline{?}, \underline{?})$

$D' = T_{-8,2}(5, 5) = (\underline{?}, \underline{?}) = (\underline{?}, \underline{?})$

To use matrices, construct the translation matrix by showing the point (–8, 2) in each of four columns.

$$\overset{T_{-8,2}}{\begin{bmatrix} -8 & -8 & \underline{?} & \underline{?} \\ 2 & 2 & \underline{?} & \underline{?} \end{bmatrix}} + \overset{\begin{matrix} Q & U & A & D \end{matrix}}{\begin{bmatrix} 1 & 6 & \underline{?} & \underline{?} \\ 1 & 2 & \underline{?} & \underline{?} \end{bmatrix}} = \overset{\begin{matrix} Q' & U' & A' & D' \end{matrix}}{\begin{bmatrix} -7 & -2 & \underline{?} & \underline{?} \\ 3 & 4 & \underline{?} & \underline{?} \end{bmatrix}}$$

b. *Q′U′A′D′* is the image of *QUAD* under a translation 8 units to the left and 2 units up. Copy and complete the graph shown above.

STOP **QY**

QY

Calculate the slopes of $\overleftrightarrow{QU}$ and $\overleftrightarrow{Q'U'}$. What is the relationship between $\overleftrightarrow{QU}$ and $\overleftrightarrow{Q'U'}$?

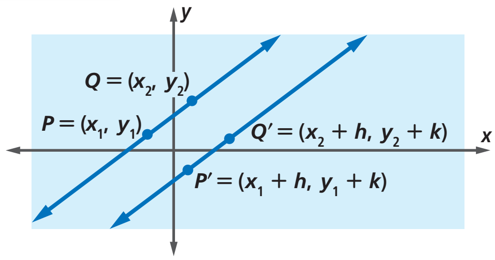
Properties of Translation Images

The QY illustrates a special case of the following more general result.

Parallel Lines and Translations Theorem

Under a translation, a preimage line is parallel to its image.

Proof Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two different points on the line $\overleftrightarrow{PQ}$. The image of the line under $T_{h,k}$ contains the points P' and Q' such that $P' = (x_1 + h, y_1 + k)$ and $Q' = (x_2 + h, y_2 + k)$.



Case 1: $\overleftrightarrow{PQ}$ is a *vertical* line. Then $x_1 = x_2$, and so $x_1 + h = x_2 + h$.

From this, $\overleftrightarrow{P'Q'}$ is also a vertical line. Thus, in this case $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$.

Case 2: $\overleftrightarrow{PQ}$ is *not vertical*. Then $x_1 \neq x_2$, and both $\overleftrightarrow{PQ}$ and $\overleftrightarrow{P'Q'}$ have slopes.

Let $m_1 =$ slope of $\overleftrightarrow{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$.

Let $m_2 =$ slope of $\overleftrightarrow{P'Q'} = \frac{(y_2 + k) - (y_1 + k)}{(x_2 + h) - (x_1 + h)} = \frac{y_2 - y_1}{x_2 - x_1}$.

The slopes are equal. So, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$.

This theorem lets you easily find an equation for a translation image of a line.

Architect Frank Lloyd Wright used many parallel lines in designing his houses as seen in his Robie House above.

GUIDED

Example 2

Find the image of the line with equation $y = 3x - 5$ under the translation $T_{2,-1}$.

Solution Because the image is parallel to the original line, the slopes are equal. So, the slope of the image is _?_.

Pick a point on the original line. An easy one is (0, –5). Its translation image is $T_{2,-1}(0, -5) = ($_?_, _?_$)$.

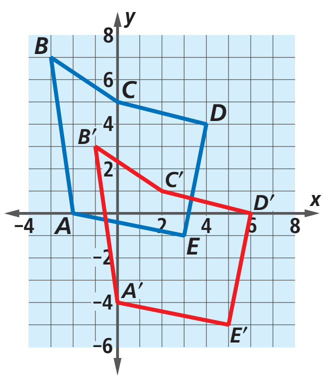
Using the Point-Slope Theorem, an equation for the line is $y =$ _?_$(x -$ _?_$) +$ _?_.

So an equation of the line is $y =$ _?_$x +$ _?_.

Questions

COVERING THE IDEAS

1. Refer to $\triangle ABC$ and $\triangle A'B'C'$ at the beginning of the lesson. Using $f(x)$ notation, describe the translation that maps
   a. $\triangle ABC$ onto $\triangle A'B'C'$.
   b. $\triangle A'B'C'$ onto $\triangle ABC$.
2. **Fill in the Blank** A translation $T_{h,k}$ is a transformation mapping (x, y) onto ___?___.
3. **Fill in the Blanks** $T_{h,k}$ is a translation of ___?___ units horizontally and ___?___ units vertically.
4. Find the image of the point under $T_{5,-7}$.
   a. (0, 0)
   b. (-50, 83)
   c. (a, b)
5. Consider $\triangle PQR$ represented by the matrix $\begin{bmatrix} 4 & 12 & -5 \\ -3 & 0 & 7 \end{bmatrix}$. Use matrix addition to find a matrix for $\triangle P'Q'R'$, the image of $\triangle PQR$ under a translation 7 units to the left and 2 units down.
6. The matrix $\begin{bmatrix} 7 & 2 & 1 & -1 & 4 \\ 2 & 13 & 8 & 4 & -5 \end{bmatrix}$ represents pentagon $AHMED$.
   a. Apply the translation $T_{-3,8}$ to the pentagon. Call the image $A'H'M'E'D'$.
   b. Graph the preimage and the image on the same set of axes.
   c. Verify that $AA' = HH'$.
   d. Why is the result in Part c not a surprise?
7. Refer to Example 1.
   a. What is the slope of $\overleftrightarrow{QA}$?
   b. What is the slope of $\overleftrightarrow{Q'A'}$?
   c. Is $\overleftrightarrow{QA} \parallel \overleftrightarrow{Q'A'}$? Justify your answer.
8. Suppose lines ℓ_1 and ℓ_2 are not parallel. Can they be translation images of each other? Explain your reasoning.
9. Refer to the graph at the right.
   a. What translation maps $ABCDE$ onto $A'B'C'D'E'$?
   b. Verify that $\overline{BC} \parallel \overline{B'C'}$.
   c. Verify that $BC = B'C'$.
   d. $BB'C'C$ is what kind of quadrilateral?



10. Consider the line with equation $y = -2x + 7$. Find an equation for the image of this line under $T_{-4,5}$.

APPLYING THE MATHEMATICS

11. Under $T_{-2,5}$ the image of $\triangle CUB$ is $\triangle C'U'B'$. $\triangle C'U'B'$ is then translated under $T_{7,3}$ to get $\triangle C''U''B''$.

a. What single translation will give the same result as $T_{7,3} \circ T_{-2,5}$?

b. Is $T_{7,3} \circ T_{-2,5} = T_{-2,5} \circ T_{7,3}$?

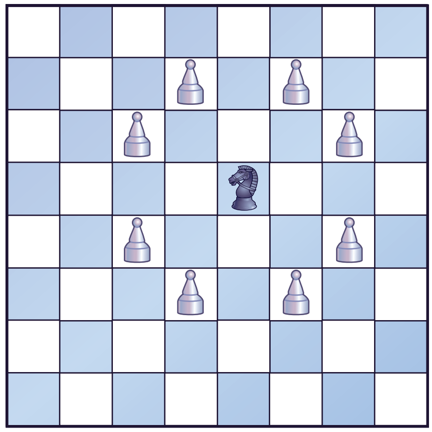
c. In general, is the composition of two translations commutative? Why or why not?

12. Line ℓ has the equation $y = 3x - 7$. Line ℓ' is the image of ℓ under a translation.

a. If ℓ' contains the point (0, 5), find an equation for ℓ'.

b. Give an example of a translation that maps line ℓ onto line ℓ'.

13. In chess, each knight can move 2 squares vertically or horizontally and then one square at a right angle to the first part of its move. In the figure at the right, the black knight can move to any of the places occupied by a white pawn. As a translation, two of the knight's possible moves can be written as $T_{2,1}$ (two squares right and one square up) and $T_{-1,2}$ (one square left and two squares up). The knight has 6 other possible moves; write them as translations.



14. Consider the segments $\overline{QU}$ and $\overline{Q'U'}$ from Example 1.

a. Find the lengths of both segments. How do they compare?

b. Do you think it is generally true that a segment and its image under a translation are congruent? If so, try to prove it.

c. Are there any other transformations under which a segment and its image are congruent? If so, which ones?

15. a. Find the image of the ordered pair (x, y) under $T_{h,k} \circ R_{90}$ and $R_{90} \circ T_{h,k}$.

b. Is translating a rotation image of a point the same as rotating its translation image? Why or why not?

REVIEW

16. Let $A = (2, 1)$ and $B = (-4, 4)$. Let $A' = R_{90}(A)$ and $B' = R_{90}(B)$. **(Lessons 4-9, 4-8)**

a. Find the coordinates of A' and B'.

b. Find equations for $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$.

c. Find the slopes of the two lines and verify that their product is -1.

17. A rotation of 180° can be considered as the composite of two rotations of 90° or as the composite of a reflection over the x-axis followed by a reflection over the y-axis. **(Lessons 4-7, 4-6, 4-3)**

  a. Compute $N \cdot N$, where $N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the matrix for R_{90}.

  b. Compute $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Do you get the same answer as in Part a?

18. Consider the line with equation $2x - 3y = 12$. **(Lessons 4-6, 3-3)**

  a. Find the x- and y-intercepts of this line.

  b. Find the images of the x- and y-intercept points under R_{90}.

  c. Write an equation in standard form for the line that contains the two points from Part b.

19. The Central High School debate team is selling T-shirts to raise money. They are selling small, medium, and large shirts in blue, grey, and tan. **(Lessons 4-2, 4-1)**

  a. The debate team ordered 8 of each color in small, 12 of each in medium, and 8 of each in large. Write a matrix to represent their T-shirt inventory.

  b. The first five customers bought 2 medium blue T-shirts, 1 medium grey T-shirt, 1 large grey T-shirt, and 1 small tan T-shirt. Write a matrix to represent these T-shirt purchases.

  c. Use matrix subtraction to write a matrix that represents the T-shirt inventory after the first five purchases.

This T-shirt has a logo that can be read both as printed and when it is rotated 180°.

20. At Mimi's Pizzeria, a large pizza costs \$8, and each additional topping costs \$0.90. Write an equation describing the total cost y of a large pizza with x toppings. Graph the resulting line, and label the y-intercept. **(Lesson 3-1)**

EXPLORATION

21. Using translations you can find a rule for finding image points for a rotation of 90° with any point as center. Consider the rotation of 90° with center $(-3, -1)$. Let (x, y) be any point.

  a. What is the image of (x, y) under the translation that maps $(-3, -1)$ onto $(0, 0)$?

  b. What is the image of your answer to Part a under a rotation of 90° about the origin?

  c. What is the image of your answer to Part b under the translation that maps $(0, 0)$ onto $(-3, -1)$?

  d. Check that your answer to Part c is the image of (x, y) under a rotation of 90° about $(-3, -1)$.

QY ANSWER

The slope of each line is $\frac{1}{5}$. They are parallel.