Lesson

4-9 Rotations and Perpendicular Lines

BIG IDEA If a line has slope m, any line perpendicular to it has slope $-\frac{1}{m}$.

If a rotation of magnitude 90° is applied to a line, then the image line is perpendicular to the preimage line. This is true regardless of the center of the rotation, and it explains a very nice relationship between the slopes of perpendicular lines.

Mental Math

Pete has $9.50 in nickels and dimes. How many dimes does he have if he has

a. 50 nickels?

b. 150 nickels?

c. n nickels?

Activity

MATERIALS matrix polygon application

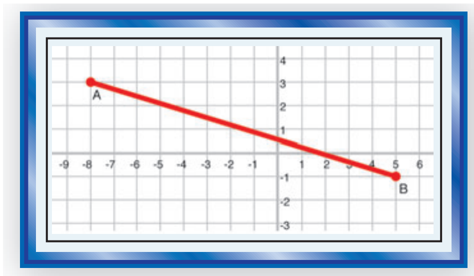
Consider $\overline{AB}$ with endpoints $A = (-8, 3)$ and $B = (5, -1)$.

Step 1 Use a matrix polygon application to enter W, the 2×2 matrix of endpoints of $\overline{AB}$. Graph $\overline{AB}$.

Step 2 Enter the R_{90} transformation matrix as R.

Step 3 Calculate $R \cdot W$ to graph $\overline{A'B'}$, the image of $\overline{AB}$ under R_{90}. Describe any relationships you notice between these two line segments.

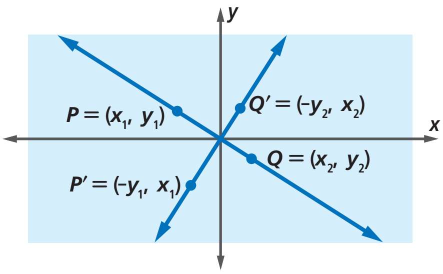
Step 4 Calculate the slope of $\overline{AB}$ and the slope of $\overline{A'B'}$. How is one slope related to the other?



The results of the Activity can be generalized to prove the following theorem.

Perpendicular Lines and Slopes Theorem (Part 1)

If two lines with slopes m_1 and m_2 are perpendicular, then $m_1 \cdot m_2 = -1$.



Proof You are given two perpendicular lines with slopes m_1 and m_2 and need to show that the product m_1m_2 is –1. The given lines either contain the origin or they are parallel to lines that contain the origin. Here we prove the theorem for two lines through the origin. In the next lesson you will show that this property holds true for perpendicular lines elsewhere.

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on a line $\overleftrightarrow{PQ}$ that contains the origin.

From Lesson 4-8, $R_{90}(P) = (-y_1, x_1) = P'$, and $R_{90}(Q) = (-y_2, x_2) = Q'$. $\overleftrightarrow{P'Q'}$ is perpendicular to $\overleftrightarrow{PQ}$ since $\overleftrightarrow{P'Q'}$ is the image of $\overleftrightarrow{PQ}$ under a rotation of magnitude 90°.

Now, let the slope of the preimage be m_1 and the slope of the image be m_2.

$$m_1 = \text{slope of } \overleftrightarrow{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_2 = \text{slope of } \overleftrightarrow{P'Q'} = \frac{x_2 - x_1}{-y_2 - (-y_1)} = \frac{x_2 - x_1}{-(y_2 - y_1)} = -\frac{x_2 - x_1}{y_2 - y_1}$$

The product of the slopes is

$$m_1 \cdot m_2 = \frac{y_2 - y_1}{x_2 - x_1} \cdot \left(-\frac{x_2 - x_1}{y_2 - y_1}\right) = -1.$$

This proves the theorem.

STOP **QY1**

QY1

Two perpendicular lines have slopes $\frac{1}{2}$ and s. What is s?

Do you see why the product of the slopes is –1? The rotation of 90° switches x- and y-coordinates and changes the first coordinate to its opposite. In the slope formula, this switches numerator and denominator and multiplies the denominator by –1. So the slope $\frac{a}{b}$ becomes $\frac{-b}{a}$, and the product is –1.

GUIDED

Example

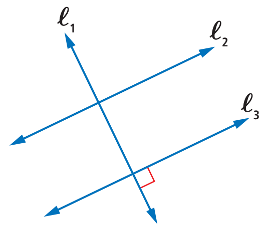
Line n contains (4, –3) and is perpendicular to line ℓ, whose equation is $y = \frac{2}{5}x + 3$. Find an equation for line n.

Solution The slope of line ℓ is ___?___. So by the theorem on the previous page, the slope of line n is ___?___.

Since line n contains (4, –3), an equation for line n in point-slope form is ___?___.

Consider the converse of the preceding theorem. Suppose line ℓ_1 has slope m_1 and line ℓ_2 has slope m_2, and $m_1m_2 = -1$. Are ℓ_1 and ℓ_2 always perpendicular? The answer is yes.

Proof We want to show that ℓ_1 and ℓ_2 are perpendicular. Think of a third line ℓ_3 with slope m_3 in the same plane as ℓ_1 and ℓ_2, and with $\ell_3 \perp \ell_1$. Then $m_1m_3 = -1$. So, $m_1m_3 = m_1m_2$. Thus, $m_3 = m_2$, so ℓ_3 and ℓ_2 have the same slope. So, $\ell_3 \parallel \ell_2$. But $\ell_1 \perp \ell_3$. Now use the theorem from geometry that if a line is perpendicular to one of two parallel lines, it must be perpendicular to the other. So, $\ell_1 \perp \ell_2$. This proves the converse of the previous theorem.



Perpendicular Lines and Slopes Theorem (Part 2)

If two lines have slopes m_1 and m_2 and $m_1m_2 = -1$, then the lines are perpendicular.

Because the original theorem and its converse are both true, we can conclude the following biconditional.

Perpendicular Lines and Slopes Theorem

Two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1m_2 = -1$.

STOP QY2

> **QY2**
> Are the lines with slopes 5 and −0.2 perpendicular? Justify your answer.

Questions

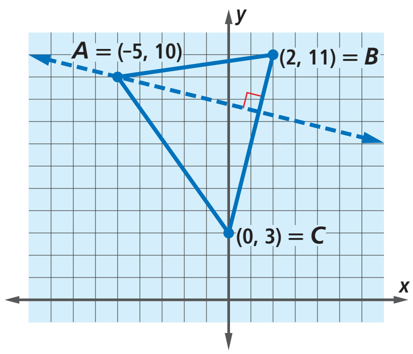
COVERING THE IDEAS

1. Write a 2×2 matrix that represents the line containing the points (x_1, y_1) and (x_2, y_2).
2. Let $\overleftrightarrow{AB}$ contain points $A = (4, 7)$ and $B = (-3, 5)$.
   a. Find two points on the image of $\overleftrightarrow{AB}$ under R_{90}.
   b. Graph $\overleftrightarrow{AB}$ and its image $\overleftrightarrow{A'B'}$.
   c. Find the slopes of $\overleftrightarrow{AB}$ and of $\overleftrightarrow{A'B'}$.
   d. What is the product of the slopes?

In 3–5, indicate whether each statement is *true* or *false*.

3. If two lines have slopes m_1 and m_2, and $m_1 m_2 = -1$, then the lines are perpendicular.
4. If two lines are perpendicular, and they have slopes m_1 and m_2, then $m_1 m_2 = -1$.
5. Suppose m_1 and m_2 are the slopes of two perpendicular lines. Then m_1 is the reciprocal of m_2.
6. A line has slope 0.25. What is the slope of a line
   a. parallel to this line?
   b. perpendicular to this line?
7. Find an equation of the line through (6, 1) and perpendicular to the line with equation $y = \frac{3}{5}x - 2$.
8. Find an equation of the line perpendicular to the line with equation $y - 2 = -5(x - 1)$ passing through the point (–10, 7).

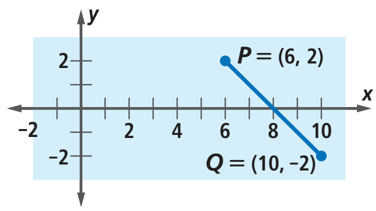
APPLYING THE MATHEMATICS



9. Use the graph at the right. Find an equation for the line perpendicular to $\overline{BC}$ through point A.
10. A line ℓ_1 contains the points (5, –3) and (9, –1).
    a. Graph ℓ_1 and write a matrix $M1$ representing this line.
    b. Find $M2 = R_{90}(M1)$ and graph the line ℓ_2 that $M2$ represents. Are ℓ_1 and ℓ_2 perpendicular?
    c. Let matrix $M3 = R_{90}(M2)$ represent line ℓ_3. What is the relationship between ℓ_1 and ℓ_3? Explain your answer.
11. Why do the statements of the theorems in this lesson apply only to lines with nonzero slopes?
12. **Multiple Choice** What is the slope of a line perpendicular to the line with equation $x = 5$?

    A 0 B The slope is not defined.

    C $-\frac{1}{5}$ D 5
13. Find an equation for the line through (6, 3) and perpendicular to the line with equation $y = 8$.
14. Refer to the graph at the right. Find an equation for the perpendicular bisector of $\overline{PQ}$. (*Hint:* First find the midpoint of $\overline{PQ}$.)



15. Let $A = (6, 2)$ and $B = (-5, 0)$.

a. **Fill in the Blank** A counterclockwise rotation of 270° is the same as a clockwise rotation of __?__.

b. Find the coordinates of A' and B', the images of A and B under R_{270}.

c. Find the slopes of $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$.

d. What relationship exists between the slopes? What does this tell you about the lines?

16. Consider $ABCD$, where $A = (0, 5)$, $B = (4, 3)$, $C = (1, -2)$, and $D = (-4, -1)$. Use the Perpendicular Lines and Slopes Theorem to determine if $ABCD$ is a rectangle.

Fill in the Blanks In 17–20, assume all lines lie in the same plane.

a. Fill each blank with $\parallel$ or $\perp$.

b. Draw a picture to illustrate each situation.

17. If $\ell \parallel m$ and $m \parallel n$, then ℓ __?__ n.

18. If $\ell \parallel m$ and $m \perp n$, then ℓ __?__ n.

19. If $\ell \perp m$ and $m \parallel n$, then ℓ __?__ n.

20. If $\ell \perp m$ and $m \perp n$, then ℓ __?__ n.

21. If M is a matrix representing a line that does not contain the origin, tell if each operation on M gives a matrix representing a line that is parallel to M, perpendicular to M, or neither perpendicular nor parallel to M.

a. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} M$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M$

c. $3M$

d. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} M$

e. $\begin{bmatrix} 3 & 3 \\ -5 & -5 \end{bmatrix} + M$

f. $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} M$

22. Without doing any computation, find $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^4$. Explain how you know your answer is true.

Figures and their rotation images can be found in Hawaiian quilts, batik designs, and Celtic knots.

REVIEW

23. Find a matrix for $R_{80} \circ R_{80} \circ R_{200}$. **(Lesson 4-8)**

24. **a.** Use a DGS to rotate the points (1, 0) and (0, 1) 80° about the origin. Find the coordinates of their image points.
    b. Use the results of Part a to find an approximate matrix for R_{80}. **(Lessons 4-8, 4-6)**

25. **Multiple Choice** Which of the following is *not* a property of multiplication of 2×2 matrices? **(Lesson 4-7)**
    A associativity
    B commutativity
    C existence of an identity
    D closure

26. Architects designing auditoriums use the fact that sound intensity I is inversely proportional to the square of the distance d from the sound source. **(Lessons 2-3, 2-2)**
    a. Write a variation equation that represents this situation.
    b. A person moves to a seat that is 4 times as far from the sound source. How will the intensity of the sound be affected?

Sydney Opera House

EXPLORATION

27. **a.** Consider the lines with equations $\pi x + \sqrt{10}y = \sqrt[3]{6}$ and $\sqrt{10}x + \pi y = \sqrt[3]{6}$. How is the graph of one related to the graph of the other?
    b. Generalize Part a.

QY ANSWERS

1. $s = -2$

2. Yes; because $5(-0.2) = -1$, the lines are perpendicular.