Lesson 4-8 Matrices for Rotations

Vocabulary

rotation
center of a rotation
magnitude of a rotation

BIG IDEA Matrices can represent rotations about the origin.

Recall from geometry and Lesson 4-7 that a turn, or **rotation**, is described by its *center* and *magnitude*. The **center of a rotation** is a point that coincides with its image. The **magnitude of a rotation** is positive if the rotation is counterclockwise, while the magnitude is negative if the rotation is clockwise. The rotation of magnitude $x°$ around the origin is denoted R_x.

Note that we denote a rotation with a capital R, while we identify a reflection with a lowercase r.

Mental Math

Solve:

a. $V = \frac{1}{3}\pi r^2 h$ for h.

b. $h = 4.9t^2$ for t.

c. $M = \frac{kwt^2}{d}$ for k.

d. $y = 4x + 12$ for x.

The Composite of Two Rotations

Rotations often occur one after the other, as when a spinner is spun twice.

Activity

MATERIALS tracing paper, protractor

At the right, figures I, II, and III are images of each other under rotations with center *O*.

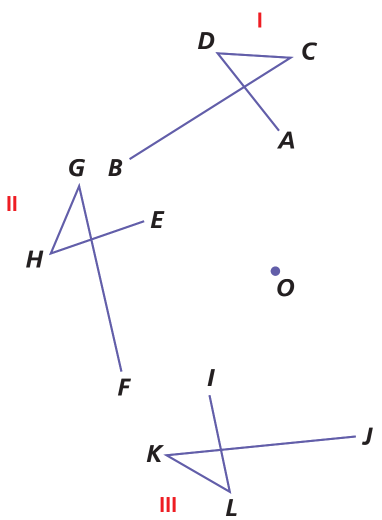
Step 1 Trace the figures at the right.

Step 2 Measure $\angle AOE$ to determine the magnitude of the rotation that maps figure I onto figure II.

Step 3 Measure $\angle EOI$ to determine the magnitude of the rotation that maps figure II onto figure III.

Step 4 What is the magnitude of the rotation that maps figure I onto figure III?

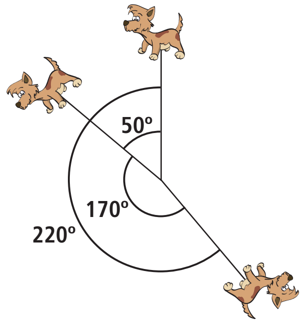
Step 5 What is the magnitude of the rotation that maps figure III onto figure I?



The relationships in the Activity result from a fundamental property of rotations, which itself is a consequence of the Angle Addition Postulate in geometry.

Composite of Rotations Theorem

A rotation of $b°$ following a rotation of $a°$ with the same center results in a rotation of $(a + b)°$. In symbols, $R_b \circ R_a = R_{a+b}$.



Matrices for Rotations

Rotations centered at the origin are the only rotations that can be represented by 2×2 matrices. This is because any transformation represented by a 2×2 matrix maps (0, 0) onto itself: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

As you will see later in this course, you can use trigonometry to develop a rotation matrix of any magnitude. However, some rotations have matrices whose elements can be derived without trigonometry. For example, in the previous lesson you derived a matrix for R_{90}.

Matrix for R_{90} Theorem

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the matrix for R_{90}.

You can verify this theorem using the Matrix Basis Theorem.

The image of (1, 0) under R_{90} is (0, 1).

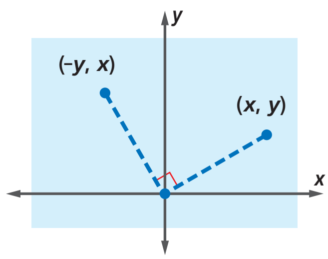
The image of (0, 1) under R_{90} is (-1, 0).

So, the matrix for R_{90} is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Using matrices to represent rotations lets you describe rotation images algebraically. For example, for R_{90},

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \cdot x + -1 \cdot y \\ 1 \cdot x + 0 \cdot y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Thus, $R_{90}(x, y) = (-y, x)$. This is an important result you should memorize.



STOP QY1

QY1

What is the image of (3, 2) under R_{90}?

Matrices for Rotations of Multiples of 90°

The rotations R_{180} and R_{270} are especially important, and their matrices can be computed straightforwardly from the matrix for R_{90}.

GUIDED

Example

Find the matrix for

a. R_{180}. b. R_{270}.

Solution

a. Because $90° + 90° = 180°$, a rotation of 180° can be considered as the composite of a 90° rotation following a first 90° rotation. That is, $R_{180} = R_{90} \circ R_{90}$.

Use matrix multiplication:

The matrix for $R_{180} = R_{90} \circ R_{90}$ is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \underline{?} & \underline{?} \\ \underline{?} & \underline{?} \end{bmatrix}$.

Another way to write the matrix for R_{180} is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2$.

b. $R_{270} = R_{90} \circ R_{\underline{?}}$.

So, the matrix for R_{270} is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \underline{?} & \underline{?} \\ \underline{?} & \underline{?} \end{bmatrix} = \begin{bmatrix} \underline{?} & \underline{?} \\ \underline{?} & \underline{?} \end{bmatrix}$.

STOP QY2

QY2

What is the matrix for R_{-90}?

Questions

COVERING THE IDEAS

1. A rotation with negative magnitude represents a turn in which direction?

2. a. Verify that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   b. What does this computation prove about representing rotations with 2×2 matrices?

3. a. **Fill in the Blank** A rotation of 45° followed by a rotation of 90° is equivalent to a rotation of __?__.
   b. Write Part a using R notation.

4. A rotation of –120° is the same as a rotation of what positive magnitude?

5. Write the matrices for R_{90}, R_{180}, R_{270}, and R_{360}.

6. Find the image of (x, y) under R_{270}.

7. Find the image of the square $\begin{bmatrix} 2 & 8 & 8 & 2 \\ 3 & 3 & 9 & 9 \end{bmatrix}$ under R_{180}.

APPLYING THE MATHEMATICS

8. Let $A = (6, 2)$, $B = R_{90}(A)$, $C = R_{180}(A)$, and $D = R_{270}(A)$. Prove that $ABCD$ is a square by showing that $AB = BC = CD = AD$ and $AO = BO = CO = DO$, where $O = (0, 0)$.

9. a. Calculate a matrix for $R_{90} \circ r_x$.
   b. Calculate a matrix for $r_x \circ R_{90}$.
   c. Your answers to Parts a and b should be different. What property of matrix multiplication does this illustrate? What property of transformations does this illustrate?

10. The matrix for R_{60} is approximately $\begin{bmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{bmatrix}$. Use the fact that a 150° rotation is the result of a 60° rotation followed by a 90° rotation to find an approximate matrix for R_{150}.

11. a. Find a matrix for $R_{90} \circ S_{0.5,2} \circ R_{-90} \circ S_{2,0.5}$.
    b. What does this transformation represent geometrically?

REVIEW

12. a. By what matrix can you multiply the size change matrix $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$ to get the identity matrix?
    b. What does your answer to Part a represent geometrically? **(Lessons 4-7, 4-4)**

13. **True or False** The matrix for a transformation can be determined by finding the image of the point (1, 1) under the transformation. **(Lesson 4-6)**

14. Consider the triangle $\triangle ELK$ represented by the matrix $\begin{bmatrix} 0 & 8 & 9 \\ -2 & -4 & -6 \end{bmatrix}$, and the triangle $\triangle RAM$ represented by the matrix $\begin{bmatrix} 0 & 4 & 4.5 \\ -3 & -6 & -9 \end{bmatrix}$. **(Lesson 4-5)**

a. Find a matrix for the scale change that maps $\triangle ELK$ onto $\triangle RAM$.

b. Find a matrix for the scale change that maps $\triangle RAM$ onto $\triangle ELK$.

15. In 2006, about 45 million people in the U.S. did not have health insurance. The four states with the highest percent of uninsured residents were Texas (24.1%), New Mexico (21.0%), Florida (20.3%), and Arizona (19.0%). Their populations in 2006 were as follows: Texas 23,508,000; New Mexico 1,955,000; Florida 18,090,000; and Arizona 6,166,000. Use matrix multiplication to determine how many people in these four states had no health insurance. **(Lessons 4-3, 4-1)**

The population of New Mexico increased by 20.1% between 1990 and 2000.

EXPLORATION

16. You can find a matrix for R_{45} in the following way. Suppose the matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, since $R_{45} \circ R_{45} = R_{90}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

a. Multiply the matrices on the left side of the equation to obtain one matrix.

b. Equate the elements of the product matrix and the matrix for R_{90}. You will have four equations in a, b, c, and d.

c. From the equations, explain why $a^2 = d^2$, but $a \neq -d$.

d. From the equations, explain why $b = -c$.

e. Either by hand or with a CAS, find the two possible matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

f. One solution is the matrix for R_{45}. The other solution is the matrix for what transformation?

QY ANSWERS

1. (–2, 3)

2. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$