Lesson

4-7 Transformations and Matrices

Vocabulary

composite of two transformations

BIG IDEA The product of two matrices corresponds to the composite of the transformations that the matrices represent.

Properties of 2 × 2 Matrix Multiplication

Multiplication of 2 × 2 matrices has some of the same properties as multiplication of real numbers, such as closure, associativity, and the existence of an identity.

1. Closure: *The set of 2 × 2 matrices is closed under multiplication.*
   Closure means that when an operation is applied to two elements in a set, the result is an element of the set. From the definition of multiplication of matrices, if you multiply two 2 × 2 matrices, the result is a 2 × 2 matrix.
2. Associativity: *Multiplication of 2 × 2 matrices is associative.* For any 2 × 2 matrices A, B, and C, it can be shown that $(AB)C = A(BC)$. You are asked to prove this in the Questions.
3. Identity: *The matrix* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is the identity for multiplication of 2 × 2 matrices.*

   Here is a proof of the Identity Property. Recall that for the real numbers, 1 is the identity for multiplication because for all a, $1 \cdot a = a \cdot 1 = a$. In the matrix version of this property, we need to show that for all 2 × 2 matrices M, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot M = M$, and $M \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = M$.

   Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   Then $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   Thus, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity for multiplication of 2 × 2 matrices.

STOP **QY1**

Mental Math

Azra has a collection of 35 china elephants. All the elephants are gray, white, or pink. She has twice as many pink elephants as white elephants and half as many gray elephants as white elephants.

a. How many of Azra's elephants are white?

b. How many are gray?

c. How many are pink?

QY1

Verify that both matrix products $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ equal $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Notice that both multiplications were needed in the proof because multiplication of 2×2 matrices is not always commutative.

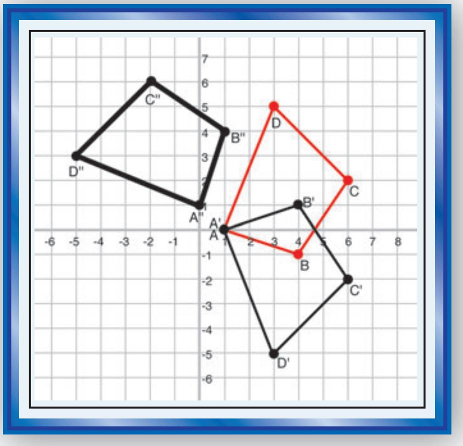
Multiplying the Matrices of Two Transformations

The product of two 2×2 transformation matrices yields a third 2×2 matrix that also represents a transformation.

Activity

MATERIALS matrix polygon application

Step 1 Use a matrix polygon application to draw quadrilateral $ABCD$ with $A = (1, 0)$, $B = (4, -1)$, $C = (6, 2)$, and $D = (3, 5)$.



Step 2 Reflect $ABCD$ over the x-axis and call its image $A'B'C'D'$.

Step 3 Reflect $A'B'C'D'$ over the line with equation $y = x$ and call its image $A''B''C''D''$.

Step 4 Compare the coordinates of the vertices of the final image $A''B''C''D''$ to the coordinates of the vertices of your original preimage $ABCD$. What single transformation could you apply to $ABCD$ to get the image $A''B''C''D''$?

Step 5 Multiply the matrix for r_x on the left by the matrix for $r_{y=x}$. Call the product M.

Step 6 Clear the screen and redraw $ABCD$. Then draw the image of $ABCD$ after applying the transformation defined by M. Compare your results to the results of Step 3. What do you notice?

Composites of Transformations

We call the final quadrilateral $A''B''C''D''$ from the Activity the image of $ABCD$ under the *composite* of the reflections r_x and $r_{y=x}$. In general, any two transformations can be composed.

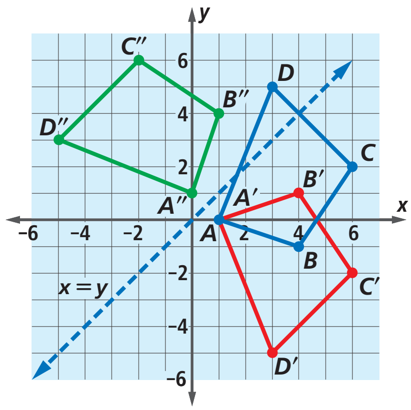
Definition of Composite of Transformations

Suppose transformation T_1 maps figure G onto figure G', and transformation T_2 maps figure G' onto figure G''. The transformation that maps G onto G'' is called the **composite** of T_1 and T_2, written $T_2 \circ T_1$.

The symbol ∘ is read "following." In the Activity, r_x came first and then $r_{y=x}$, so we write $r_{y=x} \circ r_x$ and say "$r_{y=x}$ following r_x" or "the composite of r_x and $r_{y=x}$".

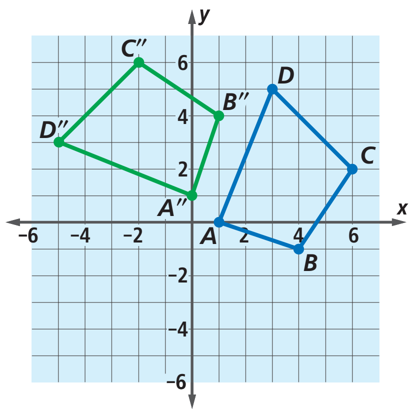
The Composite of $r_{y=x}$ and r_x

To describe $r_{y=x} \circ r_x$ as one transformation, consider the results of the previous Activity. Quadrilateral $ABCD$ is the preimage, $A'B'C'D'$ is the first image, and $A''B''C''D''$ is the final image. In Step 6 of the Activity, you plotted only the preimage $ABCD$ and the final image $A''B''C''D''$ to get a graph like the one at the right below.



The graph shows that the composite is a *rotation* 90° counterclockwise about (0, 0). We denote this rotation by R_{90}. This rotation is the composite of the two reflections: $R_{90} = r_{y=x} \circ r_x$.

In geometry, you should have encountered the Two-Reflection Theorem for Rotations: The composite of two reflections $r_m \circ r_\ell$ over intersecting lines ℓ and m is a rotation whose center is the intersection of ℓ and m. The magnitude of rotation is twice the measure of an angle formed by ℓ and m, measured from ℓ to m. Because the x-axis and the line with equation $y = x$ intersect at the origin at a 45° angle, this theorem explains why the composite $r_{y=x} \circ r_x$ is a rotation about the origin with magnitude 90°.



Matrices and Composites of Transformations

How can you find the matrix associated with R_{90}? In Steps 5 and 6 of the Activity, you multiplied the matrices for the two reflections, then multiplied the matrix for $ABCD$ on the left by the product. The result was the matrix for $A''B''C''D''$. In symbols, we can show this as follows.

$$\overset{r_{y=x}\ \circ\ r_x}{\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)} \overset{(A\ \ B\ \ C\ \ D)}{\begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & -1 & 2 & 5 \end{bmatrix}} =$$

$$\overset{R_{90}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \cdot \overset{(A\ \ B\ \ C\ \ D)}{\begin{bmatrix} 1 & 4 & 6 & 3 \\ 0 & -1 & 2 & 5 \end{bmatrix}} = \overset{A''\ \ B''\ \ C''\ \ D''}{\begin{bmatrix} 0 & 1 & -2 & -5 \\ 1 & 4 & 6 & 3 \end{bmatrix}}$$

This tells us that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a matrix for R_{90}. The general idea is summarized in the following theorem.

Matrices and Composites Theorem

If $M1$ is the matrix for transformation T_1, and $M2$ is the matrix for transformation T_2, then $M2M1$ is the matrix for $T_2 \circ T_1$.

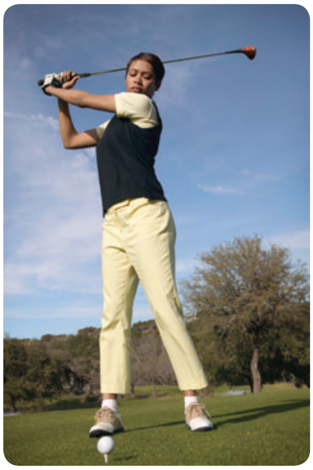
STOP QY2

QY2

If M1 represents $r_{y=x}$ and M2 represents r_x, does $M1M2 = M2M1$? Why or why not?

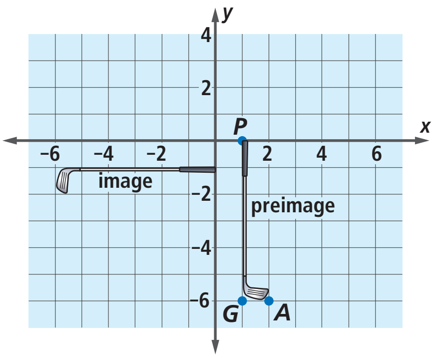
Example

Golfers strive to achieve the elusive perfect swing. Assume that when looking at a right-handed golfer from the front, as a golfer begins the swing, the golf club rotates clockwise about a point near the tip of the shaft. The club starts from a vertical position at setup, then moves along a 90° arc until it is horizontal at some point during the backswing. The rotation is R_{-90}. For the left-handed golfer, the rotation will be counterclockwise, or R_{90}. The preimage golf club and the image golf club are pictured in the graph below at the right. Three points on the original golf club are $P = (1, 0)$, $G = (1, -6)$, and $A = (2, -6)$. Use matrix multiplication to find the image of PGA under R_{-90}.



Solution A counterclockwise rotation of 90° is a composite of two reflections given by $R_{90} = r_{y=x} \circ r_x$.

To find R_{-90}, a rotation 90° clockwise about (0, 0), you need to measure the angle formed in the opposite direction, that is, the angle formed by the line with equation $y = x$ and the x-axis, measured from $y = x$ to the x-axis. So reverse the order of composition: $R_{-90} = r_x \circ r_{y=x}$.



So the matrix for R_{-90} is $\underset{r_x}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} \underset{r_{y=x}}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \underset{R_{-90}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$.

To find the image of golf club PGA, multiply its matrix by the matrix for R_{-90}.

$$R_{-90}(PGA) = \underset{R_{-90}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \underset{(P\ G\ A)}{\begin{bmatrix} 1 & 1 & 2 \\ 0 & -6 & -6 \end{bmatrix}} = \underset{P'\ G'\ A'}{\begin{bmatrix} 0 & -6 & -6 \\ -1 & -1 & -2 \end{bmatrix}}$$

Questions

COVERING THE IDEAS

1. a. When a 2 × 2 matrix is multiplied by a 2 × 2 matrix, what are the dimensions of the product matrix?
   b. What property of 2 × 2 matrix multiplication does this demonstrate?
2. What is the *multiplicative identity* for 2 × 2 matrices?

3. Write this product as a single matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ \sqrt{8} & \pi \end{bmatrix}$.

4. **Fill in the Blank** The identity transformation maps each point of a preimage onto __?__.

5. In general, multiplication of 2×2 matrices is not commutative, but sometimes it is.

   a. Suppose I is the 2×2 identity matrix and A is any other 2×2 matrix of your choosing. Does $IA = AI$?

   b. Pick any 2×2 size change matrix A and any 2×2 matrix B other than the identity matrix. Does $AB = BA$?

   c. Give an example of two 2×2 matrices A and B such that $AB \neq BA$.

Multiple Choice **In 6–10, identify which property is illustrated by the statement.**

A closure B associativity C existence of an identity

D commutativity in some cases E noncommutativity

6. $\begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}\left(\begin{bmatrix} 0 & 3 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix}\right) = \left(\begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 8 & 6 \end{bmatrix}\right)\begin{bmatrix} 5 & 9 \\ 1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 8 & 6 \end{bmatrix} \neq \begin{bmatrix} 0 & 3 \\ 8 & 6 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 7 & 4 \end{bmatrix}$

9. The product of any 2×2 matrix multiplication is a 2×2 matrix.

10. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

11. Noel thought that by applying the Associative Property, the following matrix equation would be true.

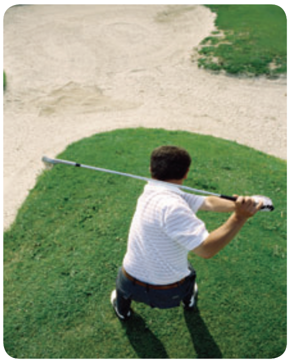
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix}\right) = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

However, when checking the matrix multiplication with a calculator, the left side of the equation did not equal the right side of the equation. Explain the mistake in Noel's logic.

12. If T_1 and T_2 are two transformations, what does $T_1 \circ T_2$ mean?

13. Does $r_{y=x} \circ r_x = r_x \circ r_{y=x}$? Explain why or why not.

14. **Fill in the Blanks** R_{-90} represents a rotation of __?__ degrees about __?__ in a __?__ direction.



15. Refer to the Example. As a right-handed golfer takes the club back the golfer cocks his or her wrist at the top of the backswing as shown in the picture. When the golfer is viewed from the back, the club appears to have rotated 270 degrees (R_{270}) from vertical.

a. Use the Matrix Basis Theorem to find the matrix for R_{270}.

b. Find the image of the golf club under a 270° counterclockwise rotation.

c. Compare the results of Part b to the results of the Example. Explain your findings.

APPLYING THE MATHEMATICS

16. Let $G = (-1, 4)$, $R = (-3, 6)$, $E = (2, 6)$, $A = (2, -1)$, and $T = (-1, -4)$.

a. Write the matrix for $GREAT$.

b. Write the matrix for $R_{90}(GREAT)$.

c. Write the matrix for $r_{y=x} \circ R_{90}(GREAT)$.

d. What single transformation is equal to $r_{y=x} \circ R_{90}$?

17. Refer to the Activity and quadrilateral $ABCD$ with $A = (1, 0)$, $B = (4, -1)$, $C = (6, 2)$, and $D = (3, 5)$.

a. Find the image of $ABCD$ under the transformation $S_{\frac{1}{2}} \circ r_y$.

b. Is the image congruent to $ABCD$?

c. Is the image similar to $ABCD$?

18. a. To what single transformation is each of the following equivalent?

i. $r_x \circ r_x$ **ii.** $r_y \circ r_y$ **iii.** $r_{y=x} \circ r_{y=x}$

b. Explain the geometric meaning of the results of Part a.

19. Consider the matrix multiplications shown at the right. Laura hypothesizes that reversing the order of a 2×2 matrix multiplication always results in switching the diagonal elements of the original 2×2 product matrix. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

$$\begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 18 & 47 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 47 & 18 \\ 30 & 11 \end{bmatrix}$$

a. Find AB and BA.

b. Is Laura's conjecture correct? Why or why not?

20. a. Use a CAS to show that

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\begin{bmatrix} i & j \\ k & \ell \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} i & j \\ k & \ell \end{bmatrix}\right).$$

b. What property does Part a demonstrate?

REVIEW

21. Name three points that are their own images under the transformation represented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. **(Lesson 4-6)**

22. Suppose T is a transformation with T: $(1, 0) \rightarrow (5, 0)$, and T: $(0, 1) \rightarrow \left(0, \frac{4}{5}\right)$. **(Lessons 4-6, 4-5)**

a. Find a matrix that represents T.

b. What kind of transformation is T?

23. Find a matrix for the size change that maps the triangle $\begin{bmatrix} -6 & 4 & 7 \\ 2 & 2 & 8 \end{bmatrix}$ onto the triangle $\begin{bmatrix} -9 & 6 & 10.5 \\ 3 & 3 & 12 \end{bmatrix}$. **(Lesson 4-4)**

24. a. Suppose a is a real number. For what values of a is there a unique line containing $(a, 0)$ and $(0, a)$?

b. If a is such that there is a unique line, find an equation satisfied by every point (x, y) on it. **(Lesson 3-4)**

25. There are two prices for used movies at the Lights, Camera, Action! video store. The regular price is \$15, but some are on sale for \$9. Let s represent the number of movies Bob bought on sale, and r represent the number of movies he bought at regular price. **(Lesson 3-2)**

a. Suppose Bob spent \$78 on movies. Write an equation relating s, r, and the money Bob spent.

b. Graph the equation from Part a and list all possible combinations of regular and sale movies Bob could have bought.

EXPLORATION

26. Test whether the properties of 2×2 matrix multiplication extend to 3×3 matrices.

a. Explain whether you think that the closure property holds for 3×3 matrix multiplication.

b. There is also an identity matrix M for 3×3 matrix multiplication. Use your knowledge of matrices to guess the values of the elements of matrix M. Test your identity matrix M by multiplying it by a general 3×3 matrix.

c. Test the commutative property for 3×3 matrix multiplication with a specific example.

d. Test the associative property for 3×3 matrix multiplication with a specific example.

QY ANSWERS

1. $\begin{bmatrix} 1 \cdot a + 0 \cdot c & 1 \cdot b + 0 \cdot d \\ 0 \cdot a + 1 \cdot c & 0 \cdot b + 1 \cdot d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $\begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2. No; $M1M2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $M2M1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The matrices are not equal.