Lesson 4-6 Matrices for Reflections

Vocabulary

reflection image of a point over a line
reflecting line, line of reflection
reflection

▶ **BIG IDEA** Matrices can represent reflections.

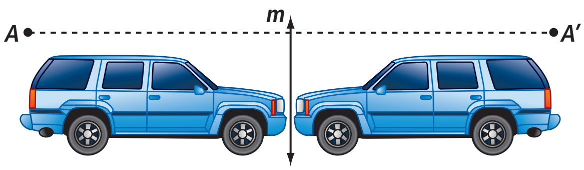
What Is a Reflection?

Recall from geometry that the **reflection image of a point *A* over a line *m*** is:

1. the point A, if A is on m;
2. the point A' such that m is the perpendicular bisector of $\overline{AA'}$, if A is not on m.

The line m is called the **reflecting line** or **line of reflection**.

A **reflection** is a transformation that maps a figure to its reflection image. The figure on the right is the *reflection image* of a drawing and the point A over the line m. This transformation is called r_m, and we write $A' = r_m(A)$.



Mental Math

Use matrices *A*, *B*, and *C* below. Tell whether it is possible to calculate each of the following.

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}$$

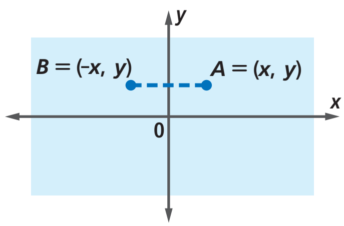
$$B = \begin{bmatrix} 4 & 0.5 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 3 \\ 0 & -12 \end{bmatrix}$$

a. $A + B$
b. AB
c. BA
d. BC

Reflection over the *y*-axis

Suppose that $A = (x, y)$ and $B = (-x, y)$, as shown at the right. Notice that the slope of $\overline{AB}$, like the slope of the x-axis, is zero, which means that $\overline{AB}$ is parallel to the x-axis, and perpendicular to the y-axis. Also, because the y-coordinates are the same and the x-coordinates are opposites, the points are equidistant from the y-axis. So the y-axis is the perpendicular bisector of $\overline{AB}$. This means that $B = (-x, y)$ is the reflection image of $A = (x, y)$ over the y-axis. Reflection over the y-axis can be denoted $r_{y\text{-axis}}$ or r_y. In this book we use r_y.



You can write

$$r_y: (x, y) \rightarrow (-x, y) \text{ or } r_y(x, y) = (-x, y).$$

Both are read "the reflection over the y-axis maps point (x, y) onto point $(-x, y)$."

Notice that $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$.

This means that there is a matrix associated with r_y and proves the following theorem.

Matrix for r_y Theorem

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is the matrix for r_y.

Example 1

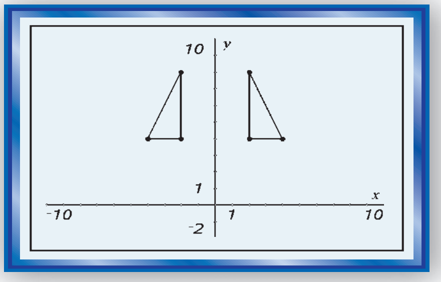
If $J = (1, 4)$, $K = (2, 4)$, and $L = (1, 7)$, find the image of $\triangle JKL$ under r_y.

Solution Represent r_y and $\triangle JKL$ as matrices and multiply.

$$\underset{r_y}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{\triangle JKL}{\begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 7 \end{bmatrix}} = \underset{\triangle J'K'L'}{\begin{bmatrix} -1 & -2 & -1 \\ 4 & 4 & 7 \end{bmatrix}}$$

The image $\triangle J'K'L'$ has vertices $J' = (-1, 4)$, $K' = (-2, 4)$, and $L' = (-1, 7)$.

Check Use a DGS to plot $\triangle JKL$ and its image $\triangle J'K'L'$. The preimage and image are graphed at the right. It checks.



Remembering Transformation Matrices

You have now seen matrices for size changes, scale changes, and one reflection, r_y. You may wonder: Is there a way to generate a matrix for any transformation A so that I do not have to just memorize them? One method of generating transformation matrices that works for reflections, rotations, and scale changes is to use the following two-step algorithm.

Step 1 Find the image of (1, 0) under A and write the coordinates in the first column of the transformation matrix.

Step 2 Find the image of (0, 1) under A and write these coordinates in the second column.

For example, here is a way to remember the matrix for r_y, the reflection over the y-axis.

The image of (1, 0) under r_y is (–1, 0).

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The image of (0, 1) under r_y is (0, 1).

This general property is called the *Matrix Basis Theorem*.

Matrix Basis Theorem

Suppose A is a transformation represented by a 2×2 matrix. If $A : (1, 0) \rightarrow (x_1, y_1)$ and $A : (0, 1) \rightarrow (x_2, y_2)$, then A has the matrix $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$.

Proof Let the 2×2 transformation matrix for A be $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose

$A: (1, 0) \rightarrow (x_1, y_1)$ and $A: (0, 1) \rightarrow (x_2, y_2)$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}.$$

Multiply the 2×2 matrices on the left side of the equation.

$$\begin{bmatrix} a \cdot 1 + b \cdot 0 & a \cdot 0 + b \cdot 1 \\ c \cdot 1 + d \cdot 0 & c \cdot 0 + d \cdot 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

Thus, the matrix for the transformation A is $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$.

Reflections over Other Lines

Other transformation matrices let you easily reflect polygons over lines other than the *y*-axis.

GUIDED

Example 2

Use the Matrix Basis Theorem to find the transformation matrix for $r_{y=x}$.

Solution Find the image of (1, 0) under $r_{y=x}$ and write its coordinates in the first column of the matrix for $r_{y=x}$.

$r_{y=x}(1, 0) = (\underline{\ ?\ }, \underline{\ ?\ })$

$$\begin{bmatrix} \underline{\ ?\ } & \underline{\ ?\ } \\ \underline{\ ?\ } & \underline{\ ?\ } \end{bmatrix}$$

Find the image of (0, 1) under $r_{y=x}$ and write its coordinates in the second column of the matrix for $r_{y=x}$.

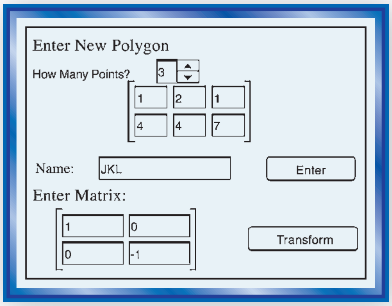
$r_{y=x}(0, 1) = (\underline{\ ?\ }, \underline{\ ?\ })$

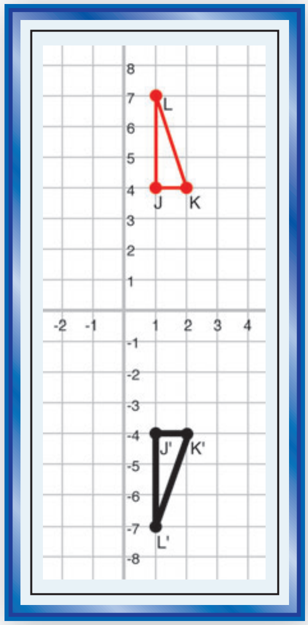
$\begin{bmatrix} \underline{\ ?\ } & \underline{\ ?\ } \\ \underline{\ ?\ } & \underline{\ ?\ } \end{bmatrix}$ is the matrix for $r_{y=x}$.

Activity

MATERIALS matrix polygon application

Step 1 Familiarize yourself with how a matrix polygon application lets you draw and transform polygons.





Step 2 Enter the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 7 \end{bmatrix}$, which represents $\triangle JKL$.

Step 3 Enter the transformation matrix $T1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

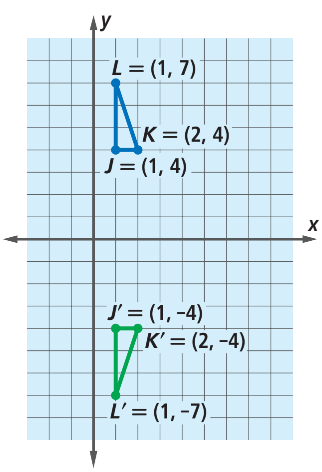
Step 4 Graph $\triangle J'K'L'$, the image of $\triangle JKL$ under the transformation represented by $T1$. How do the coordinates of corresponding points on $\triangle J'K'L'$ and $\triangle JKL$ compare? What transformation maps $\triangle JKL$ onto $\triangle J'K'L'$? If you are not sure, enter another polygon and transform it.

Step 5 Enter two more transformation matrices,

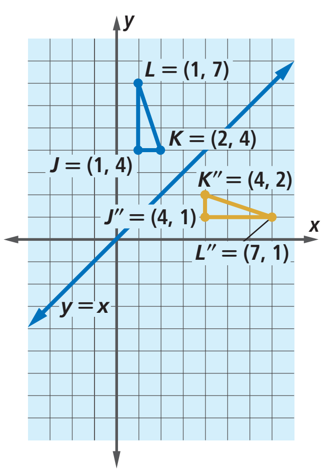
$$T2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } T3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Repeat Step 4, answering the same questions for each of these transformations.

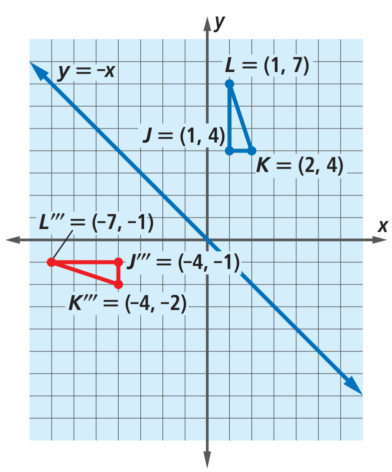
The Activity verifies that matrix multiplication can be used to reflect $\triangle JKL$ over three lines: the x-axis, the line $y = x$, and the line $y = -x$. Graphs of the triangle and its images are shown below.



$r_x(\triangle JKL) = \triangle J'K'L'$



$r_{y=x}(\triangle JKL) = \triangle J''K''L''$



$r_{y=-x}(\triangle JKL) = \triangle J'''K'''L'''$

In general, in mapping notation,

r_x: $(x, y) \rightarrow (x, -y)$, $r_{y=x}$: $(x, y) \rightarrow (y, x)$, and $r_{y=-x}$: $(x, y) \rightarrow (-y, -x)$.

It is easy to prove that the matrices for r_x, $r_{y=x}$, and $r_{y=-x}$ are as stated in the next theorem.

Matrices for r_x, $r_{y=x}$, and $r_{y=-x}$ Theorem

1. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the matrix for r_x.
2. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the matrix for $r_{y=x}$.
3. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is the matrix for $r_{y=-x}$.

Proof of 1

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x + -1 \cdot y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

You are asked to prove 2 and 3 in Questions 14 and 15.

STOP **QY**

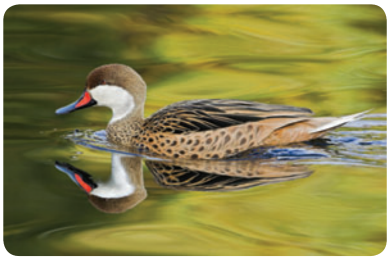
You have seen that matrix multiplication can be used to perform geometric transformations such as size changes, scale changes, and reflections. It is important to note how reflections, size changes, and scale changes differ. All reflections preserve shape and size, so reflection images are always congruent to their preimages. All size-change images are similar to their preimages, but only S_1 and S_{-1} yield congruent images. In general, scale-change images are neither congruent nor similar to their preimages.

QY

Let $A = (1, 3)$, $B = (4, 5)$, and $C = (-2, 6)$. Use matrix multiplication to find the image of $\triangle ABC$ under r_x.

Questions

COVERING THE IDEAS



1. Consider the photograph of a duck and its reflection image shown at the right. For the preimage, assume that the coordinates of the pixel at the tip of the bill are (–30, 150). What are the coordinates of the pixel at the tip of the bill of the image if the reflecting line is the x-axis?
2. Sketch $\triangle DEF$ with $D = (0, 0)$, $E = (0, 3)$ and $F = (4, 0)$. Sketch its reflection image $\triangle D'E'F'$ over the y-axis on the same grid.

3. **Fill in the Blank** Suppose that A' is the reflection image of A over m, and A' is not on line m. Then m is the perpendicular bisector of __?__.

4. What is the reflection image of a point C over a line m if C is on m?

5. Write in symbols in two ways: "The reflection over the x-axis maps point (x, y) onto point $(x, -y)$."

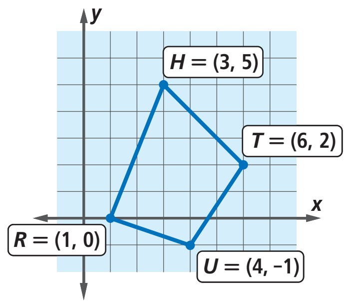
6. Translate "$r_{y=x}(x, y) = (y, x)$" into words.

Multiple Choice In 7–9, choose the matrix that corresponds to the given reflection.

A $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ B $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ C $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ D $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ E $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

7. $r_{y=x}$ 8. r_x 9. r_y

10. a. Write a matrix for quadrilateral $RUTH$ shown at the right.
    b. Use matrix multiplication to draw $R'U'T'H'$, its reflection image over the line with equation $y = -x$.



True or False In 11 and 12, if the statement is false, provide a counterexample.

11. Reflection images are always congruent to their preimages.

12. Reflection images are always similar to their preimages.

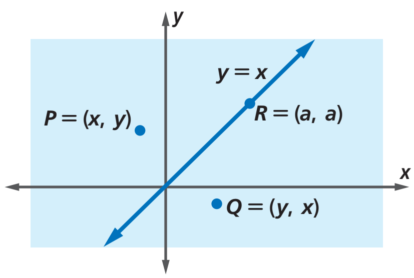
13. **Fill in the Blanks** The matrix equation $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ shows that the reflection image of the point __?__ over the line with equation __?__ is the point __?__.

APPLYING THE MATHEMATICS

14. Prove that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a matrix for the reflection over the line with equation $y = x$.

15. Prove that $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is a matrix for the reflection over the line with equation $y = -x$.

16. Suppose $P = (x, y)$ and $Q = (y, x)$. Let $R = (a, a)$ be any point on the line with equation $y = x$.
    a. Verify that $PR = QR$. (*Hint:* Use the Pythagorean Distance Formula.)
    b. **Fill in the Blank** Therefore, the line with equation $y = x$ is the __?__ of $\overline{PQ}$.
    c. What does your answer to Part b mean in terms of reflections?



REVIEW

17. a. What is the image of $(2, -3)$ under $S_{0.5,\,2}$?

b. Write a matrix for $S_{0.5,\,2}$.

c. Describe $S_{0.5,\,2}$ in words. (Lesson 4-5)

18. When does the scale change matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ represent a size change? (Lessons 4-5, 4-4)

19. Let $P = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, and $Q = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. (Lesson 4-4)

a. Compute the distance between P and Q.

b. Compute the distance between P' and Q' if

$$P' = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} P, \text{ and } Q' = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} Q.$$

c. Does the transformation given by the matrix $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ preserve distance?

20. a. Write the first 6 terms of the sequence defined by

$$\begin{cases} c_1 = 1 \\ c_n = (-1)^n c_{n-1} - 3, \text{ for integers } n \geq 2 \end{cases}.$$ (Lesson 3-6)

b. Rewrite the second line of this recursive definition if the previous term is called c_n.

21. The surface area A of a box with length ℓ, height h, and width w is given by the equation $A = 2\ell h + 2\ell w + 2hw$. (Lesson 1-7)

a. Rewrite this equation if all edges are the same length, x.

b. Solve the equation in Part a for x.

22. On the television show *The Price is Right*, contestants spinning the Big Wheel have to make sure it turns at least one full revolution. Describe the range in degrees d that the Wheel must turn in order for the spin to count. (Previous course)

EXPLORATION

23. a. Either by graphing manually, or by using a DGS, find the matrix for the reflection $r_{y=2x}$.

b. To check this matrix, test at least five points. Make sure that at least two of the points are on the line of reflection.

QY ANSWER

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 & -2 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 \\ -3 & -5 & -6 \end{bmatrix};$$

$A' = (1, -3)$, $B' = (4, -5)$, and $C' = (-2, -6)$