Lesson 4-5 Matrices for Scale Changes

Vocabulary

scale change
horizontal magnitude
vertical magnitude
stretch
shrink

BIG IDEA Matrices can represent scale changes.

What Is a Scale Change?

In the previous lesson you studied size changes, which are transformations in which the changes to the preimage have the same magnitude in both the horizontal and vertical directions. Sometimes it is more useful to apply a transformation in which the horizontal and vertical changes have *different* magnitudes.

For example, imagine that a friend hands you a rubber band and says there is a secret message written on it. The writing on the rubber band looks like the picture at the left below. If you apply a size change, and stretch the rubber band by the same amount in both directions, it looks like the picture in the center. It is still unreadable. However, if you stretch the rubber band more in the horizontal direction than the vertical direction, it looks like the picture at the right, and you can read the message.

Transformations that multiply coordinates by constants are called *scale changes*.

Definition of Scale Change

For any nonzero numbers a and b, the transformation that maps (x, y) onto (ax, by) is called the **scale change** with **horizontal magnitude** a and **vertical magnitude** b, and is denoted $S_{a,b}$.

$$S_{a,b}(x, y) = (ax, by) \text{ or}$$

$$S_{a,b} : (x, y) \rightarrow (ax, by)$$

When $|a| > 1$ (or $|b| > 1$), the scale change is a **stretch** in the horizontal (or vertical) direction. When $|a| < 1$ (or $|b| < 1$), the scale change is a **shrink** in the horizontal (or vertical) direction.

Mental Math

Identify the quadrants in which the graph of the equation appears.

a. $y = 7x^2$

b. $y = -7x$

c. $y = -\frac{7}{x^2}$

d. $y = \frac{7}{x}$

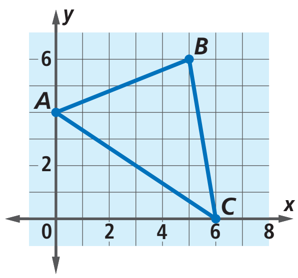
Souvenirs are often scale models of the actual things they represent.

Like size changes, scale changes are functions because each point in the preimage maps to one and only one point in the image.

GUIDED

Example 1

Consider the triangle at the right. Find its image under $S_{4,0.5}$.



Solution $S_{4,0.5}(x, y) = (\underline{\ ?\ }x, \underline{\ ?\ }y)$, so multiply all x-coordinates of the preimage points by $\underline{\ ?\ }$ and all y-coordinates of the preimage points by $\underline{\ ?\ }$. Images of the vertices determine the image polygon.

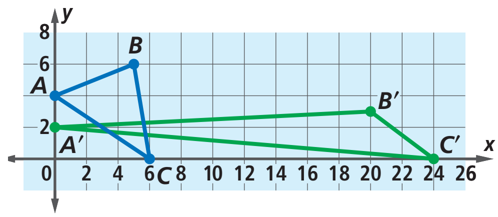
A': $S_{4,0.5}$ (0, 4) = (0, 2)

B': $S_{4,0.5}$ (5, 6) = ($\underline{\ ?\ }$, $\underline{\ ?\ }$)

C': $S_{4,0.5}$ (6, 0) = ($\underline{\ ?\ }$, $\underline{\ ?\ }$)

The image of △*ABC* is △*A′B′C′* with the vertex coordinates above.

Check Check using the graph at the right. The image of every point should be 4 times as far from the y-axis and half as far from the x-axis as its preimage. For instance, the image $B' = (20, 3)$ is 4 times as far from the y-axis and half as far from the x-axis as its preimage $B = (5, 6)$.



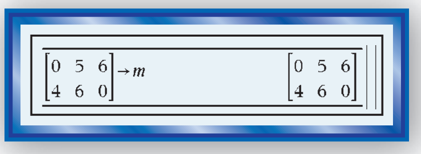
Example 1 demonstrates that scale changes might not multiply both vertical and horizontal distances by a constant amount, and so images are not necessarily similar to their preimages.

Matrices for Scale Changes

Because a size change is represented by a matrix, it is reasonable to expect that a scale change has a matrix.

Activity

Materials CAS or graphing calculator



Step 1 Sketch a graph of △*ABC* from Example 1. Write a matrix to represent △*ABC* and store the matrix as *M* on a CAS.

Step 2 Define a second matrix *S*1 on your CAS, with $S1 = \begin{bmatrix} 4 & 0 \\ 0 & 0.5 \end{bmatrix}$. Calculate $S1 \cdot M$, the matrix for △*A′B′C′*.

Step 3 Sketch △*A′B′C′* on the same graph. Compare your graph to the graphs in Example 1. What transformation do you think matrix *S*1 represents?

Step 4 Define a third matrix $S2 = \begin{bmatrix} 1.2 & 0 \\ 0 & 3 \end{bmatrix}$. Calculate $S2 \cdot M$, the matrix for △*A″B″C″*. Sketch △*A″B″C″* on the same graph.

Step 5 Compare the coordinates and graph of $\triangle ABC$ to the coordinates and graph of $\triangle A''B''C''$. What transformation do you think *S*2 represents?

Step 6 Generalize your findings. Make a conjecture about the matrix for the scale change $S_{a,b}$.

Algebra easily proves the results of the Activity. Suppose that $S_{a,b}$ has the matrix

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

where *e, f, g,* and *h* are real numbers. Because $S_{a,b}$ maps (x, y) onto (ax, by), we want to find *e, f, g,* and *h* so that for all *x* and *y*,

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}.$$

By matrix multiplication, $\begin{bmatrix} ex + fy \\ gx + hy \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$.

Thus for all *x* and *y*, $ex + fy = ax$, so $e = a$ and $f = 0$, and $gx + hy = by$, so $g = 0$ and $h = b$. We have proved the following theorem.

Scale Change Theorem

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is the matrix for $S_{a,b}$.

STOP **QY**

QY

What is the matrix for $S_{7,4}$?

Example 2

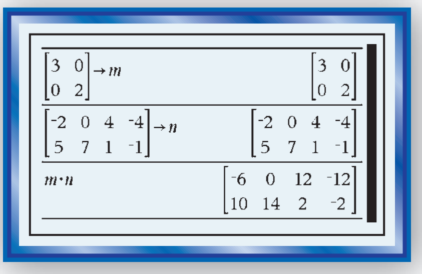
Consider the quadrilateral *ABCD* with $A = (-2, 5)$, $B = (0, 7)$, $C = (4, 1)$, and $D = (-4, -1)$. Find its image $A'B'C'D'$ under $S_{3,2}$.

Solution 1 Write $S_{3,2}$ and *ABCD* in matrix form. Calculate the product by hand.

$$\overset{S_{3,2}}{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}} \overset{\quad A \quad B \quad C \quad D}{\begin{bmatrix} -2 & 0 & 4 & -4 \\ 5 & 7 & 1 & -1 \end{bmatrix}} = \overset{\quad A' \quad B' \quad C' \quad D'}{\begin{bmatrix} -6 & 0 & 12 & -12 \\ 10 & 14 & 2 & -2 \end{bmatrix}}$$

Solution 2 Use technology to do the calculations.

Let *m* represent the transformation matrix and *n* represent quadrilateral *ABCD.*



Notice that a scale change may or may not stretch or shrink by different factors in the horizontal and vertical directions. If the factors are the same in both directions, then the scale-change matrix has the form $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, and the transformation is just a size change. Conversely, a size change with magnitude k is a scale change with horizontal magnitude k and vertical magnitude k. Thus *size changes are special types of scale changes*. In symbols, $S_k = S_{k,k}$.

Questions

COVERING THE IDEAS

1. **Fill in the Blank** $S_{0.8, \frac{4}{7}} : (x, y) \rightarrow$ __?__

2. **Multiple Choice** Which of the following mappings gives a horizontal shrink and a vertical stretch?

   A $S_{\frac{1}{5}, \frac{2}{3}}$ B $S_{9, 0.75}$ C $S_{0.36, 1.03}$ D $S_{5, 5}$

3. a. What is the image of (3, 7.5) under $S_{4, 2}$?

   b. Describe $S_{4,2}$ in words.

4. a. Draw $\triangle FLY$ with $F = (-2, 3)$, $L = (3, 1)$, and $Y = (1, 0)$.

   b. Draw its image $\triangle F'L'Y'$ under $S_{3,1}$.

   c. Which component changed, the horizontal or vertical?

In 5 and 6, refer to Example 2.

5. What are the coordinates of the image quadrilateral?

6. Find the image matrix of $ABCD$ under $S_{1,4}$.

7. **True or False** All scale changes produce images that are not similar to their preimages.

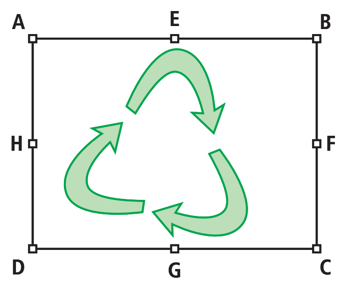
8. Describe the scale change with a horizontal shrink of magnitude $\frac{1}{8}$ and a vertical stretch of magnitude 2

   a. in $f(x)$ notation. b. in mapping notation. c. as a matrix.

9. **Fill in the Blanks** The scale change $S_{8,8}$ can be thought of as the __?__ change identified as __?__.

APPLYING THE MATHEMATICS

In 10 and 11, consider this information. Pictures in word processors can usually be resized under both scale changes and size changes. When the picture is selected, a rectangle with small boxes (handles) appears around it, similar to the one at the right. Which of the handles A through H could you move, and how would you need to move it, to apply the given transformation to the preimage at the right?



10. $S_{3, \frac{3}{4}}$

11. $S_{1.6}$

12. The transformation $S_{2,1.5}$ is applied to the rectangular fenced-in plot at the right. Coordinates are given for each vertex of the plot.

a. Find the perimeter P and area A of the pictured fenced-in plot.

b. Find the coordinates of the vertices of the image of the plot.

c. Find the perimeter P and area A of the new image.

13. $\triangle BLT$ is represented by the matrix $\begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & 0 \end{bmatrix}$.

a. Graph the triangle. Classify $\triangle BLT$ as isosceles or scalene, and as acute, right, or obtuse.

b. Find the matrix for $\triangle B'L'T'$, the image of $\triangle BLT$ under the scale change represented by $\begin{bmatrix} 4 & 0 \\ 0 & 1.2 \end{bmatrix}$.

c. Graph $\triangle B'L'T'$. Would you classify $\triangle B'L'T'$ the same way as $\triangle BLT$? If not, what is different?

14. Prove or disprove the following: If $\overline{A'B'}$ is the image of $\overline{AB}$ under a scale change, then $\overline{AB} \parallel \overline{A'B'}$.

15. Consider the matrix equation below that maps $\triangle FOR$ onto $\triangle F'O'R'$.

$$\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 25 & 5 & 20 \\ 14 & 28 & -7 \end{bmatrix}$$

a. What transformation is applied to $\triangle FOR$?

b. Find the coordinates of the vertices of $\triangle FOR$.

c. Find $\frac{F'O'}{FO}$ and $\frac{O'R'}{OR}$. (*Hint:* Use the formula for distance between two points or a DGS.)

d. Should the ratios in Part c be the same? Why or why not?

REVIEW

16. Let $R = \begin{bmatrix} 5 & -1 & -1 & 5 \\ 3 & 3 & 6 & 5 \end{bmatrix}$. **(Lesson 4-4)**

a. What matrix will produce a size change of magnitude $\frac{2}{3}$ on R?

b. Multiply your matrix in Part a by R. Call your answer R'.

c. Is the rectangle whose vertices are given by R similar to the rectangle whose vertices are given by R'? Justify your answer.

17. **Fill in the Blanks** Let $T = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, and $V = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.
(Lessons 4-4, 4-3)
   a. T is the matrix for a size change of magnitude __?__.
   b. V is the matrix for a size change of magnitude __?__.
   c. Compute TV. TV is the matrix for a size change of magnitude __?__.

18. Let $P = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. **(Lessons 4-3, 4-1, 2-7)**
   a. Compute $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}P$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}P$, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}P$, and $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}P$.
   b. If you consider all of your answers to Part a as representing points (x, y), what single direct-variation equation do x and y satisfy?

19. **Fill in the Blank** The explicit formula for the nth term of an arithmetic sequence with first term a_1 and constant difference d is __?__. **(Lesson 3-8)**

20. Suppose a line goes through the points (-4, -4) and (2, 1). **(Lesson 3-4)**
   a. Write an equation for this line in point-slope form.
   b. This line goes through a point $(7, a)$. What is the value of a?

21. a. Name three properties that are preserved under reflections.
   b. A reflection is a type of isometry. What is an isometry? Name two other types of isometries. **(Previous Course)**

QY ANSWER

$\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$

EXPLORATION

22. Let $ABCD$ be the square defined by the matrix $\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$.
   a. Transform $ABCD$ by multiplying its matrix on the left by each of the following matrices (and by some others of your own choice).
      i. $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ ii. $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$
      iii. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ iv. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$
   b. Find the area of each image. Enter your results in a table like the one shown at the right.
   c. What is the connection between the elements a and b of the scale-change matrix $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and the effect the scale change has on area?

Transformation Matrix	Preimage Area	Image Area
$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$	4 sq. units	? sq. units
$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$	4 sq. units	? sq. units
$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$	?	?
$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$	?	?