

## Lesson

## 4-4

Matrices for  
Size Changes

► **BIG IDEA**  $2 \times 2$  matrices can represent geometric transformations.

In the next few lessons, you will see how matrices can represent a variety of geometric transformations. Recall from geometry that a **transformation** is a one-to-one correspondence between the points of a **preimage** and the points of an **image**. Consider the transformation  $S_k$  that maps  $(x, y)$  onto  $(kx, ky)$ . If  $k = 5$ , the transformation images of a few points under  $S_5$  are written as follows:

$$S_5(2, -3) = (10, -15);$$

$$S_5(-3, 0) = (-15, 0);$$

$$S_5(-2, -1.5) = (-10, -7.5);$$

and, in general,  $S_5(x, y) = (5x, 5y)$ .

## Definition of Size Change

For any  $k \neq 0$ , the transformation that maps  $(x, y)$  onto  $(kx, ky)$  is called the **size change** with **center**  $(0, 0)$  and **magnitude**  $k$ , and is denoted  $S_k$ .

From the definition,  $S_k(x, y) = (kx, ky)$ ,  
or  $S_k: (x, y) \rightarrow (kx, ky)$ .

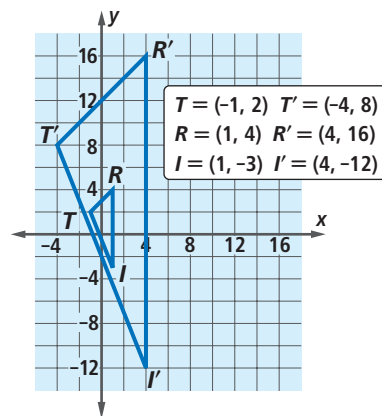
These are read “Under the size change of magnitude  $k$  and center  $(0, 0)$ , the image of  $(x, y)$  is  $(kx, ky)$ ,” or “The size change of magnitude  $k$  and center  $(0, 0)$  maps  $(x, y)$  onto  $(kx, ky)$ .”

## Example 1

- What is the magnitude of the size change that maps the preimage  $\triangle TRI$  onto its image  $\triangle T'R'I'$  as shown at the right?
- Write the size change using mapping notation.

## Solution

- Pick one coordinate pair on the preimage and compare it to the corresponding coordinate pair of the image.



## Vocabulary

transformation  
preimage  
image  
size change  
center of a size change  
magnitude of a size change  
identity matrix  
identity transformation  
similar

## Mental Math

## True or False

- All squares are rectangles.
- All rectangles are parallelograms.
- All integers are whole numbers.
- All whole numbers are integers.

$T = (-1, 2)$  and  $T' = (-4, 8)$ . The coordinates of  $T$  have been multiplied by 4. So the magnitude of the size change is 4.

- b. This size change maps  $(x, y)$  onto  $(4x, 4y)$ .  
 $S_4: (x, y) \rightarrow (4x, 4y)$

## Activity

**MATERIALS** graph paper, ruler

Work with a partner.

**Step 1** You and your partner should each copy polygon *NUMER* shown at the right, then together write a matrix to represent it.

**Step 2** One of you should multiply *NUMER* on the left by  $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ .

The other partner should multiply *NUMER* on the left by  $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ .

Each of you should draw the image polygon on your graph of *NUMER* and label it  $N'U'M'E'R'$ . Complete Steps 3–6 for your own graph.

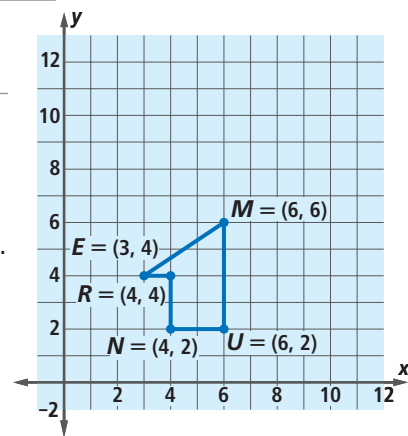
**Step 3** How do the coordinates of *NUMER* compare to the coordinates of  $N'U'M'E'R'$ ?

**Step 4** Draw a ray from the origin (point  $O$ ) through vertex  $E$  of *NUMER*. Then draw a ray from the origin through  $E'$ .

**Step 5** Measure the distances  $OE$  and  $OE'$  and compare them. What is  $\frac{OE'}{OE}$ ?

**Step 6** Repeat Steps 4 and 5 for the other vertices of *NUMER* and  $N'U'M'E'R'$ .

**Step 7** Discuss each other's results and generalize. If the matrix for *NUMER* is multiplied on the left by  $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$  to get polygon  $N'U'M'E'R'$ , where  $k$  is a real number, how do the vertices of the polygon *NUMER* compare to those of the polygon  $N'U'M'E'R'$ ?



**STOP** QY

## Using Matrices to Perform Size Changes

As the Activity illustrates, size change images can be found by multiplying matrices. When the matrix for the point  $(x, y)$  is multiplied on the left by  $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ , the matrix for the point  $(kx, ky)$  results.

**QY**

How do the lengths of sides of the polygons *NUMER* and  $N'U'M'E'R'$  from Step 2 compare?

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx + 0y \\ 0x + ky \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

This proves the following theorem.

### Size Change Theorem

$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$  is the matrix for  $S_k$ .

When  $k = 1$ , the **identity matrix**  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  maps each point  $\begin{bmatrix} x \\ y \end{bmatrix}$  of a figure onto itself.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1 \cdot y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This size change of magnitude 1 is called the **identity transformation**.

### GUIDED

#### Example 2

Using the figure from Example 1, perform the appropriate matrix multiplication to find the vertices of the image  $\triangle T'R'I'$  from the vertices of the preimage  $\triangle TRI$ .

**Solution** Write  $\triangle TRI$  and  $S_4$  in matrix form and multiply.

$$\begin{bmatrix} S_4 \\ \text{?} & 0 \\ \text{?} & 4 \end{bmatrix} \begin{bmatrix} T & R & I \\ \text{?} & 1 & \text{?} \\ 2 & \text{?} & -3 \end{bmatrix} = \begin{bmatrix} T' & R' & I' \\ -4 & \text{?} & 4 \\ \text{?} & 16 & \text{?} \end{bmatrix}$$

So  $\triangle T'R'I'$  has vertices  $T' = (-4, \text{?})$ ,  $R' = (\text{?}, 16)$ , and  $I' = (4, -12)$ .

**Check** Compare the vertices from the matrix with the graph of  $\triangle T'R'I'$ . The coordinates are the same.

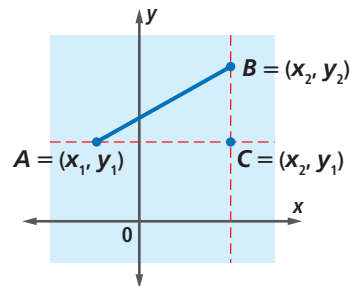
## Size Changes and the Multiplication of Distance

Recall that the Pythagorean Theorem can be applied to calculate the distance between two points given their coordinates.

Suppose you wish to find the distance  $AB$  when  $A = (x_1, y_1)$  and  $B = (x_2, y_2)$ . If you don't remember the general formula, let  $C = (x_2, y_1)$  to create right triangle  $\triangle ABC$ . In that triangle,

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$$

$\overline{AC}$  and  $\overline{BC}$  lie on horizontal and vertical lines. So their lengths are like distances on a number line.



$$AB^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

Take the square root of each side and you have the Pythagorean Distance Formula.

### Pythagorean Distance Formula

If  $A = (x_1, y_1)$  and  $B = (x_2, y_2)$ , then

$$AB = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}.$$

So, for instance, in Example 1, where  $T = (-1, 2)$  and  $I = (1, -3)$ , then

$$\begin{aligned} TI &= \sqrt{|1 - (-1)|^2 + |-3 - 2|^2} & \text{and} & \quad T'I' = \sqrt{|4 - (-4)|^2 + |-12 - 8|^2} \\ &= \sqrt{2^2 + 5^2} & & \quad = \sqrt{8^2 + 20^2} \\ &= \sqrt{29} & & \quad = \sqrt{464} \\ & & & \quad = 4\sqrt{29}. \end{aligned}$$

This example illustrates that, in the size change  $S_k$ , the distance between images is  $|k|$  times the distance between their preimages.

## Similarity and Size Changes

Size changes are fundamental in the study of similarity. Recall from geometry that two figures are **similar** if and only if one is the image of the other under a composite of reflections and size changes. Composites of reflections, such as rotations, translations, and reflections themselves, preserve distance. So, if two figures are similar, as are  $\triangle TRI$  and  $\triangle T'R'I'$  in Example 1, distances in one are a constant multiple of distances in the other.

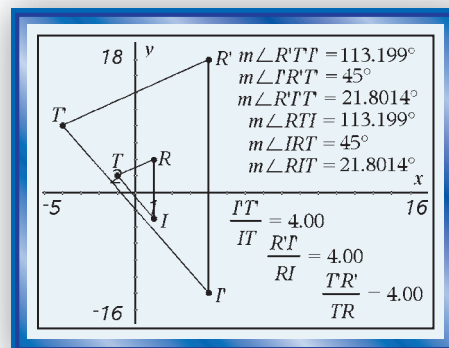
### Example 3

Refer to triangles  $\triangle TRI$  and  $\triangle T'R'I'$  in Example 1. Calculate the following to verify that they are similar.

- The ratios of each pair of corresponding sides of the triangles
- Measures of corresponding angles of the triangles

**Solution** Plot both triangles in your dynamic geometry system (DGS). Use the DGS to show measures of the angles of the triangles and the ratios of the sides. **Answers to Parts a and b** are shown in the display at the right.

Based on these measures, the triangles appear to be similar. The ratios of sides are all equal to the magnitude of the size change, 4. The corresponding angles have equal measure.



## Questions

### COVERING THE IDEAS

1. Refer to the Activity. How does  $N'U'M'E'R'$  compare to  $NUMER$  if the matrix  $NUMER$  is multiplied by  $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$  and  $k$  is:
- a. 4?                      b.  $\frac{1}{2}$ ?                      c. 1?

In 2 and 3, how is the expression read?

2.  $S_6(3, -2) = (18, -12)$                       3.  $S_{1.2}: (4, 3.4) \rightarrow (4.8, 4.08)$
4. What matrix would you use to change the size of a polygon by a magnitude of  $\frac{1}{5}$ ?
5. **Multiple Choice** If  $S_k: (2, 5) \rightarrow (1, \frac{5}{2})$ , what is the value of  $k$ ?
- A 2                      B 4                      C  $\frac{1}{2}$                       D  $\frac{1}{4}$
6. a. Write a matrix to describe the vertices of the quadrilateral  $QUAD$  with coordinates  $Q = (3, 7)$ ,  $U = (5, -9)$ ,  $A = (-2, -8)$ , and  $D = (0, 4)$ .
- b. Give the coordinates of the vertices of  $Q'U'A'D'$ , the image of the quadrilateral  $QUAD$  in Part a, under  $S_6$ .
- c. Verify that  $Q'U' = 6 \cdot QU$ .
7. Refer to Example 2. Write a size change matrix that transforms  $\triangle T'R'I'$  back to  $\triangle TRI$ .
8. **True or False** To map a point onto itself, multiply the point matrix on the left by  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

In 9–11, answer *always*, *sometimes but not always*, or *never*.

9. Under a size change, an angle and its image are congruent.
10. Under a size change, a segment and its image are congruent.
11. Under a size change, a figure and its image are similar.
12.  $\triangle ABC$  has matrix  $\begin{bmatrix} 3 & -9 & 6 \\ 8 & 12 & 4 \end{bmatrix}$ .
- a. Graph  $\triangle ABC$  and its image  $\triangle A'B'C'$  under  $S_{\frac{1}{3}}$ .
- b. Explain why  $\triangle ABC$  and  $\triangle A'B'C'$  are similar.

### APPLYING THE MATHEMATICS

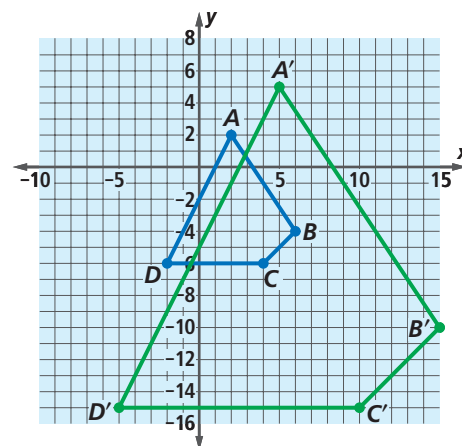
13. A Chicago souvenir store sells an exact scale replica of the Sears Tower that is 7.5 inches tall. The actual Sears Tower is 1730 feet tall. What transformation matrix could be used to change the size of the scale model to the size of the actual tower?



The Sears Tower was built in 1970–73 at a cost of \$250 million.

14. The Japanese fairy tale *Issunbōshi* (Suyeoka, Goodman, & Spicer, 1974), tells the story of a 2.5 cm tall boy named Issunbōshi who goes on a journey and eventually becomes a full-sized man.
- If the average height of a man is 170.2 cm, what size-change magnitude  $k$  is needed to transform Issunbōshi's height?
  - Issunbōshi has a cricket for a pet. The average cricket length is 3.8 cm. If the length of the cricket were transformed by the same size change as Issunbōshi, how long would the cricket be?

15. Refer to the drawing at the right.
- Find the matrix of the transformation mapping  $ABCD$  onto  $A'B'C'D'$ .
  - Find the slope of  $\overrightarrow{BC}$ .
  - Find the slope of  $\overrightarrow{B'C'}$ .
  - Is  $\overrightarrow{BC}$  parallel to  $\overrightarrow{B'C'}$ ? Why or why not?



16. Define a function pdf (Pythagorean distance function) on a CAS with inputs  $x_a$ ,  $y_a$ ,  $x_b$ , and  $y_b$  that finds the Pythagorean Distance between any two ordered pairs  $(x_a, y_a)$  and  $(x_b, y_b)$ . (We use these names because  $x_1$ ,  $y_1$ ,  $x_2$  and  $y_2$  are reserved names on many calculators.) Use pdf to check the lengths of  $TI$  and  $T'I'$  in the lesson.
17. a. Refer to Example 1. Find the image  $\Delta T^*R^*I^*$  of  $\Delta TRI$  under a size change of magnitude  $-4$ .
- b. What are some differences between the image  $\Delta T^*R^*I^*$  and the image  $\Delta T'R'I'$ ?

## REVIEW

18. **Multiple Choice** Refer to the matrices at the right. Which of the following matrix multiplications are defined? (There may be more than one correct answer.) (Lesson 4-3)

A  $AB$                       B  $AC$                       C  $BA$   
 D  $BC$                       E  $CA$                       F  $CB$

19. Let  $Q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . (Lessons 4-3, 4-1)

a. Compute  $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}Q$ ,  $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}Q$ ,  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}Q$ , and  $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}Q$ .

b. Plot  $Q$  and the four answers to Part a as points in the plane.

20. Compute  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} b & d \\ a & c \end{bmatrix} + \begin{bmatrix} d & c \\ b & a \end{bmatrix} + \begin{bmatrix} c & a \\ d & b \end{bmatrix}$ . (Lesson 4-2)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 3 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} a & b & c \\ 8 & 8 & 8 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

21. You know the volume of a rectangular solid is given by the formula  $V = \ell wh$ . (Lesson 2-9)
- Solve this equation for  $\ell$ .
  - Fill in the Blanks** From Part a,  $\ell$  varies directly as ? and inversely as ? and ?.
  - If  $w$  is multiplied by 8,  $h$  is multiplied by 9, and  $V$  is multiplied by 20, by what is  $\ell$  multiplied?

### EXPLORATION

22. Suppose  $\triangle MIA$  is represented by the matrix

$$\begin{bmatrix} 4 & 8 & 10 \\ -2 & 5 & -3 \end{bmatrix}$$

- Find the product  $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & 8 & 10 \\ -2 & 5 & -3 \end{bmatrix}$ .
- The product matrix in Part b represents  $\triangle M'I'A'$ , the image of  $\triangle MIA$  under a size change of what magnitude? Draw  $\triangle MIA$  and  $\triangle M'I'A'$ .
- You can do Parts a and b by using a matrix polygon application. First input a matrix for  $\triangle MIA$  and a matrix for a size change of a magnitude of your choosing. Then run the program for the product of those matrices.
- How are the lengths of the sides of  $\triangle M'I'A'$  related to the lengths of the sides of  $\triangle MIA$ ?
- How are the areas of  $\triangle M'I'A'$  and  $\triangle MIA$  related?

Enter New Polygon

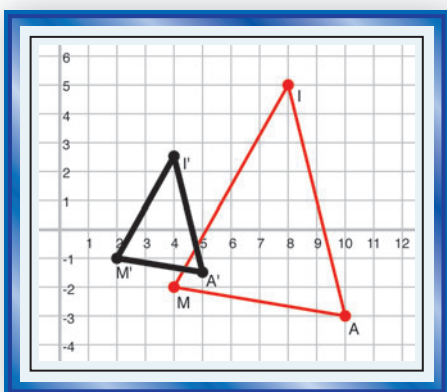
How Many Points?

|    |   |    |
|----|---|----|
| 4  | 8 | 10 |
| -2 | 5 | -3 |

Name:

Enter Matrix:

|     |     |
|-----|-----|
| 0.5 | 0   |
| 0   | 0.5 |



### QY ANSWER

The lengths of sides of  $N'U'M'E'R'$  are two times (or  $\frac{1}{3}$  of) the lengths of corresponding sides of  $NUMER$ .