Lesson

4-3 Matrix Multiplication

Vocabulary

row-by-column multiplication

matrix multiplication

matrix product

BIG IDEA Matrices with compatible dimensions can be multiplied using row-by-column multiplication.

If you want to buy four $12 pizzas, you find the total cost by multiplying: $12\ \frac{\text{dollars}}{\text{pizza}} \cdot 4$ pizzas $= \$48$. But consider a more complicated situation: the Forensics Club (F), the Jazz Band (J), and the Volleyball team (V) are each holding pizza parties, are each ordering different quantities of cheese (C), mushroom (M), and "garbage" (G) pizzas (which have everything on them), and want to compare prices at two different pizzerias, Lorenzo's and Billy's.

These data have been organized into the two matrices below. One matrix contains the prices in dollars for each type of pizza at the two pizzerias, and the second matrix contains the number of each type of pizza ordered by each club.

Price per Pizza

	C	M	G
Lorenzo's Prices	12	15	20
Billy's Prices	11	17	18

Pizzas per Club

	F	J	V
Cheese	4	1	3
Mushroom	3	5	3
Garbage	2	3	3

STOP QY1

Mental Math

Estimate the tip to the nearest dime.

a. a 10% tip on a bill of $24.95

b. a 15% tip on a bill of $32.20

c. a 20% tip on a bill of $73.83

a "garbage" pizza of vegetables

QY1

a. What is the price of a cheese pizza at Lorenzo's?

b. How many mushroom pizzas has the Forensics Club ordered?

Example 1

Write a single expression giving the total price for pizzas at Lorenzo's for the Forensics Club and evaluate it.

Solution The prices at Lorenzo's are in the first row of the Price-per-Pizza matrix. The numbers of each kind of pizza ordered by the Forensics Club are in the first column of the Pizza-per-Club matrix.

Lorenzo's Prices per Pizza

C	M	G
12	15	20

Pizzas Ordered by the Forensics Club

C	4
M	3
G	2

Total Price for the Forensics Club Order at Lorenzo's

$$\$12 \cdot 4 + \$15 \cdot 3 + \$20 \cdot 2 = \$133$$

Multiplying Two Matrices

The calculation \$12 • 4 + \$15 • 3 + \$20 • 2 = \$133 in Example 1 illustrates an idea that is used to calculate the product of these matrices. The idea is called **row-by-column multiplication**. The solution to Example 1 is shown in matrix notation below. A row matrix is multiplied by a column matrix to get a 1 × 1 product matrix.

$$\begin{bmatrix} 12 & 15 & 20 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = [12 \cdot 4 + 15 \cdot 3 + 20 \cdot 2] = [133]$$

This process can be generalized into the algorithm below.

Algorithm for Row-by-Column Multiplication

Step 1 Multiply the first element in the row matrix by the first element in the column matrix. Multiply the second element in the row matrix by the second element in the column matrix. Continue multiplying the *n*th element in the row by the *n*th element in the column until you reach the end of the row and the column.

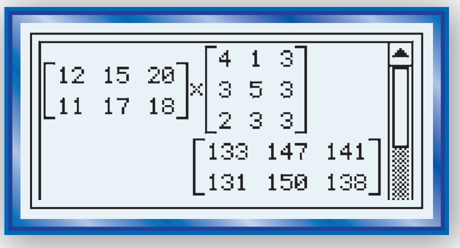
Step 2 Add the products in Step 1.

Notice that the row and the column have to have the same number of elements in order for this multiplication to work. Also, the row is on the left and the column is on the right.

The product $A \cdot B$ or AB of two matrices A and B is found by using the above algorithm to multiply each row of A times each column of B. For example, if matrix A has 2 rows and matrix B has 3 columns, then there are 6 ways to multiply a row by a column. The 6 products of these rows and columns are the 6 elements of the product matrix AB. The entry in row 2, column 3 of the product comes from multiplying row 2 of the first matrix by column 3 of the second matrix, as shown below.

$$\begin{bmatrix} 12 & 15 & 20 \\ 11 & 17 & 18 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 3 \\ 3 & 5 & 3 \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 133 & 147 & 141 \\ 131 & 150 & 138 \end{bmatrix}$$

If you multiply the Price-per-Pizza matrix by the Pizzas-per-Club matrix, you will get a product matrix for price per club. The calculator display at the right shows the product. This is not scalar multiplication; it is **matrix multiplication**.



QY2

In the 2×3 product matrix, the rows represent the 2 pizzerias and the columns represent the 3 clubs as shown at the right.

$$\begin{array}{r} \\ \text{Lorenzo's Prices} \\ \text{Billy's Prices} \end{array} \begin{array}{c} \begin{array}{ccc} \text{F} & \text{J} & \text{V} \end{array} \\ \begin{bmatrix} 133 & 147 & 141 \\ 131 & 150 & 138 \end{bmatrix} \end{array}$$

Each element in this product matrix is the total price of a pizza order. For example, the Forensics Club would pay a total of \$133 at Lorenzo's, and the Jazz Band would pay \$150 at Billy's.

QY2

Use your calculator to verify the matrix product on the previous page.

GUIDED

Example 2

Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 & 2 \\ 3 & -1 & 4 \end{bmatrix}$. Find their product AB by hand.

Solution A has _?_ rows and B has _?_ columns. So the matrix AB will have _?_ entries.

Use the algorithm for row-by-column multiplication to fill in the missing entries below.

$$AB = \begin{bmatrix} 1 \cdot 0 + 3 \cdot 3 & 1 \cdot 5 + \underline{\ ?\ } & \underline{\ ?\ } \\ -2 \cdot 0 + 4 \cdot 3 & \underline{\ ?\ } + 4 \cdot -1 & \underline{\ ?\ } \end{bmatrix} = \begin{bmatrix} 9 & \underline{\ ?\ } & \underline{\ ?\ } \\ 12 & \underline{\ ?\ } & \underline{\ ?\ } \end{bmatrix}$$

In general, to multiply matrices A and B, find all possible products using rows from matrix A and columns from matrix B. This leads to the following definition.

Definition of Matrix Multiplication

If A is an $m \times n$ matrix and B is a $n \times q$ matrix, then the **matrix product** $A \cdot B$ (or AB) is an $m \times q$ matrix whose element in row r, column c is the product of row r of A and column c of B.

Caution! The product of two matrices A and B exists only when the *number of columns of A equals the number of rows of B*. So if A is $m \times n$, B must be $n \times q$ in order for AB to exist.

These matrices can be multiplied.

$$\begin{bmatrix} 4 & -1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 & 7 \\ 5 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -11 & 28 \\ 28 & -6 & 42 \end{bmatrix}$$

2×2 $\quad 2 \times 3$ $\quad 2 \times 3$

equal

dimensions of product

These matrices cannot be multiplied.

$$\begin{bmatrix} 3 & -2 & 7 \\ 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 6 & 2 \end{bmatrix}$$

2×3 $\quad 2 \times 2$

not equal

no product

These two cases indicate that, in general, *multiplication of matrices is not commutative*. When both AB and BA exist, we can avoid confusion by saying "Multiply A on the left by B" to mean "find BA."

STOP **QY3**

When matrices arise from real situations, you often have several choices for ways of arranging your data. However, when you are multiplying two matrices, the number of columns of the first matrix must match the number of rows of the second matrix. In order to set up matrices for multiplication, think about the units involved: the *headings of the columns* of the left matrix must match the *headings of the rows* of the right matrix. That is how the matrices were set up in the opening example.

> **QY3**
> Can you multiply a 3×4 matrix on the right by a 4×2 matrix? If so, what would the dimensions of the product be? If not, why not?

Multiplying More Than Two Matrices

The Activity below suggests that an important property of real-number multiplication extends to matrix multiplication.

Activity

MATERIALS CAS or graphing calculator

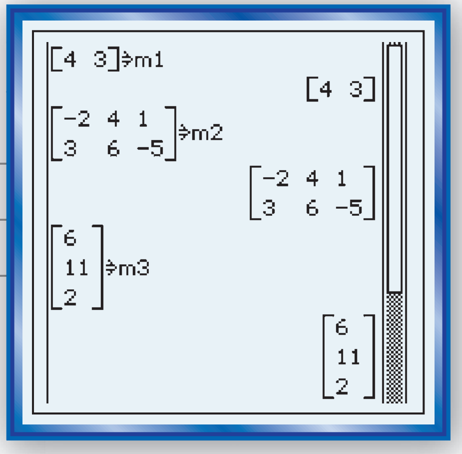
Step 1 Store these matrices in your calculator:

$M1 = \begin{bmatrix} 4 & 3 \end{bmatrix}$, $M2 = \begin{bmatrix} -2 & 4 & 1 \\ 3 & 6 & -5 \end{bmatrix}$, and $M3 = \begin{bmatrix} 6 \\ 11 \\ 2 \end{bmatrix}$.

Step 2 Calculate $(M1 \cdot M2) \cdot M3$.

Step 3 Calculate $M1 \cdot (M2 \cdot M3)$.

Step 4 Compare the results of Steps 2 and 3. What property holds true for matrices $M1$, $M2$, and $M3$?



This property holds for all matrices. For any matrices A, B, and C, if the applicable products exist, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

Matrix multiplication has many applications. In this course, you will use it to model linear combinations, to perform geometric transformations, and to solve systems of equations.

Questions

COVERING THE IDEAS

In 1–3, refer to the Price-per-Pizza, Pizzas-per-Club, and Price-per-Club matrices at the beginning of this lesson.

1. How many mushroom pizzas did the Volleyball team order?
2. How many garbage pizzas were ordered by all the clubs together?
3. Write an expression to calculate the total cost for the Jazz Band's order from Billy's pizzeria.

In 4 and 5, perform the multiplication.

4. $\begin{bmatrix} 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 4 & -5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 11 \\ 7 \end{bmatrix}$

In 6 and 7, use the matrices $A = \begin{bmatrix} 5 & 1 \\ 6 & 2 \\ 3 & -1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$.

6. a. **Fill in the Blanks** The product AB has __?__ row(s) and __?__ column(s).
   b. **Fill in the Blanks** The element in the second row and first column of AB is __?__ • __?__ + __?__ • __?__.
   c. Compute the product AB.
7. a. Is it possible to compute the product BA? Explain why or why not.
   b. What property of matrix multiplication does the answer to Part a illustrate?
8. Multiply the matrices at the right in two different ways to show that matrix multiplication is associative.

$$\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

9. Multiply the matrices at the right in two different orders to show that matrix multiplication is not commutative.

$$\begin{bmatrix} 4 & 6 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 5 & 1 \end{bmatrix}$$

10. Consider the matrices M and N below.

$$M = \begin{bmatrix} 11 & 12 & 13 & 14 & 15 \\ 21 & 22 & 23 & 24 & 25 \\ 31 & 32 & 33 & 34 & 35 \\ 41 & 42 & 43 & 44 & 45 \end{bmatrix}, N = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$$

   a. Only one of the products MN and NM exists. Which is it?
   b. What are the dimensions of the product?
   c. Write an expression for the element in the third row, second column of the product.

APPLYING THE MATHEMATICS

11. Lenora runs a small shipping company. One afternoon, she gets requests from two sporting-goods stores to transport shipments of sports equipment. Store 1 wants to transport 20 cases of table tennis balls (TTB), 50 cases of tennis balls (TB), and 12 bowling balls (BB). Store 2 wants to transport 60 cases of table tennis balls, 10 cases of tennis balls, and 15 bowling balls. One case of table tennis balls weighs 1.5 pounds, one case of tennis balls weighs 5 pounds, and one bowling ball weighs 12 pounds. In addition, each case of table tennis balls or tennis balls takes up 1 cubic foot of cargo space, but each bowling ball takes 0.6 cubic foot of space. Use matrix multiplication to find the total weight and cargo space required for each order.

Bowling balls are available in a wide variety of weights, patterns, and colors.

12. Refer to the pizza story in this lesson. According to the product matrix, is there any group that will save money by ordering their pizza at Lorenzo's? How much would the group save?

13. Consider the point matrix $A = \begin{bmatrix} x \\ y \end{bmatrix}$. Perform the indicated multiplication. Call the product B. Then describe the relationship between points A and B.

a. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot A$

b. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$

c. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot A$

d. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot A$

e. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot A$

14. You own the Sweet Cakes Bakery, which produces three sizes of muffins: mini, regular, and large. You are concerned with three main ingredients: flour, butter, and sugar. Matrix R shows the amount of each ingredient (in pounds) required per dozen of each muffin size. Matrix Q shows the average number of dozens of each muffin size sold on Mondays and Fridays.

$$\begin{array}{l} \\ \text{Flour} \\ \text{Butter} \\ \text{Sugar} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Mini} & \text{Regular} & \text{Large} \end{array} \\ \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0.1 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \end{array} = R \qquad \begin{array}{l} \\ \text{Mini} \\ \text{Regular} \\ \text{Large} \end{array} \begin{array}{c} \begin{array}{cc} \text{Mon} & \text{Fri} \end{array} \\ \begin{bmatrix} 10 & 20 \\ 15 & 35 \\ 20 & 50 \end{bmatrix} \end{array} = Q$$

a. What does the number in the third row, first column of Q represent in this situation?

b. Compute RQ.

c. What does the number in the second row, second column of RQ represent?

d. One Saturday, the weekly sugar shipment does not come in. There are 8 pounds of sugar on hand. How will that impact your preparations for Monday?

15. Suppose $\begin{bmatrix} 4 & 3 \\ 2 & x \end{bmatrix} \cdot \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} 23 \\ 13 \end{bmatrix}$. Find the value of x.

16. a. The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the *2 × 2 identity matrix.*
If $A = \begin{bmatrix} 4 & 2 \\ -3 & 6 \end{bmatrix}$, compute IA and AI.
b. Why is I called the identity matrix?

REVIEW

17. Compute $\begin{bmatrix} 6 & -4 \\ -1 & 20 \end{bmatrix} + 2\begin{bmatrix} -3 & 2 \\ \frac{1}{2} & -10 \end{bmatrix}$. **(Lesson 4-2)**

18. If $\begin{bmatrix} a & 3.3 & 5c \\ 7 & -2e & -9 \end{bmatrix} = k\begin{bmatrix} 9 & -11 & -8c \\ d & \frac{4}{9} & 5f \end{bmatrix}$, find a, c, d, e, f, and k. **(Lesson 4-2)**

19. What are the dimensions of the matrix at the right? **(Lesson 4-1)**

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & \pi \end{bmatrix}$$

20. **Multiple Choice** Which formula best models the data in the table below? **(Lesson 2-7)**
A $P = km$ B $P = km^2$ C $P = \frac{k}{m}$ D $P = \frac{k}{m^2}$

m	1	2	3	4	5	6
P	36	9	4	2.25	1.44	1

21. Let $K = (1, 3)$, $L = (-4, 3)$, $M = (-4, 8)$, and $N = (1, 8)$. **(Previous Course)**
a. What kind of figure is $KLMN$?
b. Divide each coordinate of the vertices of $KLMN$ by 2 to find the coordinates of $K'L'M'N'$.
c. What kind of figure is $K'L'M'N'$? How do you know?

22. Find the distance between each pair of points. **(Previous Course)**
a. (2, 4) and (8, 16) b. (1.2, 0.7) and (6.3, –3.75)
c. $\left(-\frac{3}{4}, \frac{2}{3}\right)$ and $\left(-\frac{4}{9}, \frac{1}{2}\right)$

EXPLORATION

23. Suppose the product of two 2 × 2 matrices A and B is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but neither matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. What is the largest number of nonzero elements possible in A and B in order for this to happen?

QY ANSWERS

1a. \$12

1b. 3

2. Calculator display should be similar to the one shown on page 236.

3. Yes; the product would be a 3 × 2 matrix.