Lesson

4-2 Matrix Addition

Vocabulary

matrix addition,
sum of two matrices

scalar multiplication,
scalar product

difference of two matrices

BIG IDEA Matrices with the same dimensions can be added in a very natural way.

How Are Matrices Added?

There are many situations which require adding the information stored in matrices. For instance, suppose matrix C represents the current inventory of cars at Rusty's Car Dealership.

$$\text{Current Inventory}$$
$$\begin{array}{r} \\ \text{Turbo} \\ \text{Cruiser} \\ \text{Clunker} \\ \text{Vacationer} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Red} & \text{Blue} & \text{White} & \text{Silver} & \text{Tan} \end{array} \\ \begin{bmatrix} 12 & 10 & 7 & 8 & 2 \\ 14 & 12 & 7 & 12 & 7 \\ 17 & 8 & 2 & 12 & 5 \\ 15 & 4 & 14 & 13 & 3 \end{bmatrix} \end{array} = C$$

A new shipment of cars arrives and the numbers stored in matrix D are the quantities of the new cars received by Rusty.

$$\text{Deliveries}$$
$$\begin{array}{r} \\ \text{Turbo} \\ \text{Cruiser} \\ \text{Clunker} \\ \text{Vacationer} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Red} & \text{Blue} & \text{White} & \text{Silver} & \text{Tan} \end{array} \\ \begin{bmatrix} 6 & 2 & 3 & 5 & 1 \\ 7 & 4 & 2 & 10 & 2 \\ 4 & 7 & 1 & 8 & 1 \\ 3 & 3 & 5 & 9 & 1 \end{bmatrix} \end{array} = D$$

The current total inventory is found by adding matrices C and D. This **matrix addition** is performed according to the following rule.

Definition of Matrix Addition

If two matrices A and B have the same dimensions, their **sum** $A + B$ is the matrix in which each element is the sum of the corresponding elements in A and B.

Add corresponding elements of C and D to find the elements of $C + D$. One set of corresponding elements is circled in the two matrices on this page and the matrix at the top of the following page: $12 + 8 = 20$.

Mental Math

At the movies, a bag of popcorn costs $4.50, a soda costs $3.25, and a package of candy costs $3.00.

a. How much do one bag of popcorn and two sodas cost?

b. How much do two sodas and two packages of candy cost?

c. How much do one bag of popcorn, one soda, and one package of candy cost?

d. How much do p bags of popcorn, s sodas, and c packages of candy cost?

Custom paint jobs such as the one above can give cars a unique look.

$$\begin{array}{r|ccccc|l} & \multicolumn{5}{c}{\text{New Inventory}} & \\ & \text{Red} & \text{Blue} & \text{White} & \text{Silver} & \text{Tan} & \\ \text{Turbo} & 18 & 12 & 10 & 13 & 3 & \\ \text{Cruiser} & 21 & 16 & 9 & 22 & 9 & \\ \text{Clunker} & 21 & 15 & 3 & 20 & 6 & = C + D \\ \text{Vacationer} & 18 & 7 & 19 & 22 & 4 & \end{array}$$

Since matrix addition involves adding corresponding elements, only matrices with the same dimensions can be added. As the car dealership example shows, the sum matrix will have the same dimensions as the matrices that were added.

Because addition of real numbers is commutative, *addition of matrices is commutative*. Thus, if two matrices A and B can be added, then $A + B = B + A$. Also, *addition of matrices is associative*, meaning that for all matrices A, B, and C with the same dimensions, $(A + B) + C = A + (B + C)$.

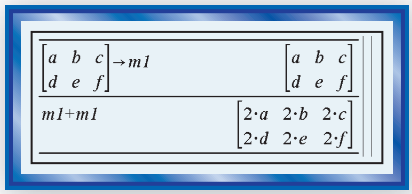
Scalar Multiplication

Matrix addition is related to a special multiplication involving matrices called *scalar multiplication*.

Activity

MATERIALS CAS

Step 1 Enter the matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ into your CAS and store it as $M1$.



Step 2 Add the matrix from Step 1 to itself and write down the result. You should see a display similar to the one at the right.

Step 3 Add the result from Step 2 to the original matrix in Step 1 and write down the result.

Step 4 If you add the result from Step 3 to the matrix in Step 1, what result do you expect? Why?

Step 5 Clear your screen and enter $2 * \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and write down the result.

Step 6 Enter $3 * \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and write down the result.

Step 7 If you enter $4 * \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, what result do you expect? Why?

Step 8: Is there a connection between your responses to Step 4 and Step 7? If so, what is it?

With real numbers, you can use multiplication as shorthand for repeated addition. For instance, $a + a + a$ can be written as $3 \cdot a$ or $3a$. The Activity gives evidence that this rule also holds true for matrices. Repeated matrix addition gives rise to an operation called **scalar multiplication**.

Definition of Scalar Multiplication

The **scalar product** of a real number k and a matrix A is the matrix kA in which each element is k times the corresponding element in A.

Example 1

Find the scalar product $6\begin{bmatrix} -2.3 & 8 \\ 7.1 & \frac{1}{2} \end{bmatrix}$.

Solution Multiply each element in the matrix by 6.

$$6\begin{bmatrix} -2.3 & 8 \\ 7.1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 6 \cdot -2.3 & 6 \cdot 8 \\ 6 \cdot 7.1 & 6 \cdot \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -13.8 & 48 \\ 42.6 & 3 \end{bmatrix}$$

How Are Matrices Subtracted?

In the set of real numbers, subtraction has the following property: $a - b = a + {-b} = a + {-1} \cdot b$. Matrices are subtracted in a similar manner:

$$A - B = A + (-1)B = A + (-B).$$

Definition of Matrix Subtraction

Given two matrices A and B with the same dimensions, their **difference** $A - B$ is the matrix in which each element is the difference of the corresponding elements in A and B.

Example 2

The matrices $P1$ and $P2$ below represent the degrees earned by men and women in four professions during 2003 and 2004.

2003

$$\begin{array}{l} \\ \text{Medicine} \\ \text{Dentistry} \\ \text{Law} \\ \text{Theology} \end{array} \begin{array}{c} \begin{array}{cc} \text{Male} & \text{Female} \end{array} \\ \begin{bmatrix} 8{,}221 & 6{,}813 \\ 2{,}653 & 1{,}691 \\ 19{,}916 & 19{,}151 \\ 3{,}499 & 1{,}852 \end{bmatrix} \end{array} = P1$$

2004

$$\begin{array}{l} \\ \text{Medicine} \\ \text{Dentistry} \\ \text{Law} \\ \text{Theology} \end{array} \begin{array}{c} \begin{array}{cc} \text{Male} & \text{Female} \end{array} \\ \begin{bmatrix} 8{,}273 & 7{,}169 \\ 2{,}532 & 1{,}803 \\ 20{,}332 & 19{,}877 \\ 3{,}511 & 1{,}821 \end{bmatrix} \end{array} = P2$$

Source: U.S. Census Bureau

a. Create a new matrix that shows the change in the number of degrees awarded from 2003 to 2004, separated by gender.

b. Which group, male or female, had the greater total increase in degrees awarded?

Solution

a. The change by gender of degrees awarded can be found by creating $P2 - P1$. Store $P1$ and $P2$ on a CAS. Then subtract the matrices as shown at the right.

$$p2-p1 \quad \begin{bmatrix} 52 & 356 \\ {}^{-}121 & 112 \\ 416 & 726 \\ 12 & {}^{-}31 \end{bmatrix}$$

b. The sum of the elements in the first column in matrix $P2 - P1$ gives the overall change for males, and the sum of the elements in the second column gives the overall change for females.

Overall change for males: $52 + {}^{-}121 + 416 + 12 = 359$

Overall change for females: $356 + 112 + 726 + {}^{-}31 = 1163$

Females had the greater total increase in degrees awarded with 1163 more degrees in 2004 than 2003.

STOP **QY**

QY

Check your answer to Part a of Example 2 by performing the subtraction of the first row by hand.

When more than one matrix operation appears in an expression, you should follow the same order of operations as with expressions involving real numbers. That is, you should perform scalar multiplication before addition or subtraction. You are asked to evaluate expressions with more than one matrix operation in Questions 4 and 5.

Questions

COVERING THE IDEAS

1. Can you add $\begin{bmatrix} 6 & -8 \end{bmatrix}$ to $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$? Explain why or why not.

In 2–5, perform the indicated operations.

2. $\begin{bmatrix} -2 & 5 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 8 & -2 \end{bmatrix}$

3. $\begin{bmatrix} 13 & -9 \\ -21 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 6 \\ -15 & 3 \end{bmatrix}$

4. $2\begin{bmatrix} -1 & -3 \\ \frac{1}{2} & \frac{5}{2} \end{bmatrix} + \begin{bmatrix} 6 & -4 \\ 2 & 0 \end{bmatrix}$

5. $-3\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -2 \end{bmatrix} - 4\begin{bmatrix} \frac{1}{4} & 0 \\ \frac{3}{4} & -1 \end{bmatrix}$

In 6 and 7, refer to Rusty's car inventory matrices *C* and *D* at the beginning of this lesson.

6. Does $C + D = D + C$? Explain why or why not.
7. Suppose matrix D represents cars in the current inventory that were damaged in a hailstorm. Find $C - D$, and explain what it represents.
8. Refer to the matrix $P2 - P1$ from Example 2.
   a. For which profession was there the greatest change in the number of degrees awarded?
   b. Is the change in Part a an increase or decrease?
9. a. **True or False** Subtraction of matrices is commutative.
   b. Let $B = \begin{bmatrix} 1 & 4 \\ 7 & 10 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 8 & 11 \end{bmatrix}$. Support your answer to Part a by evaluating $B - C$ and $C - B$.

APPLYING THE MATHEMATICS

10. The matrices $N1$, $N2$, and $N3$ give the enrollments by gender and grade at North High School over a 3-year period, beginning with $N1$ and ending with $N3$. In each matrix, row 1 gives the number of boys and row 2 the number of girls. Columns 1 to 4 give the number of students in grades 9 through 12, respectively.

Grades

$$N1 = \begin{array}{c c} \begin{array}{cccc} 9 & 10 & 11 & 12 \end{array} & \\ \begin{bmatrix} 289 & 282 & 276 & 270 \\ 299 & 288 & 276 & 264 \end{bmatrix} & \begin{array}{l} \text{boys} \\ \text{girls} \end{array} \end{array}$$

$$N2 = \begin{array}{c c} \begin{bmatrix} 240 & 228 & 216 & 204 \\ 240 & 234 & 228 & 222 \end{bmatrix} & \begin{array}{l} \text{boys} \\ \text{girls} \end{array} \end{array}$$

$$N3 = \begin{array}{c c} \begin{bmatrix} 154 & 149 & 143 & 138 \\ 143 & 143 & 138 & 132 \end{bmatrix} & \begin{array}{l} \text{boys} \\ \text{girls} \end{array} \end{array}$$

   a. Find matrix T that shows the total enrollment in each grade by gender over the 3-year time span.
   b. What was the change in enrollment for boys and girls in each grade from the first year to the last year?

11. Some key results for four teams from the National Football League for the 2005 and 2006 regular season are given in the matrices below. W = number of wins, L = number of losses, PF = points scored for the team, and PA = points scored against the team.

$$\begin{array}{cc} & \text{2006} \\ & \begin{array}{cccc} W & L & PF & PA \end{array} \\ \begin{array}{l}\text{New England}\\ \text{Pittsburgh}\\ \text{Indianapolis}\\ \text{Denver}\end{array} & \begin{bmatrix} 12 & 4 & 385 & 237 \\ 8 & 8 & 353 & 315 \\ 12 & 4 & 427 & 360 \\ 9 & 7 & 319 & 305 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \text{2005} \\ & \begin{array}{cccc} W & L & PF & PA \end{array} \\ \begin{array}{l}\text{New England}\\ \text{Pittsburgh}\\ \text{Indianapolis}\\ \text{Denver}\end{array} & \begin{bmatrix} 10 & 6 & 379 & 338 \\ 11 & 5 & 389 & 258 \\ 14 & 2 & 439 & 247 \\ 13 & 3 & 395 & 258 \end{bmatrix} \end{array}$$

Source: www.NFL.com

a. Subtract the right matrix from the left matrix. Call the difference M.

b. What is the meaning of the 4th column of M?

c. Interpret each number in the 1st row of M.

The Pittsburgh Steelers celebrated their 2006 Super Bowl win.

12. Pearl, a puzzle maker, constructs three types of animal puzzles for children in two different styles. Pearl's output last year is given in the matrix P below.

$$\begin{array}{cc} & \begin{array}{ccc} \text{Cat} & \text{Dog} & \text{Mouse} \end{array} \\ \begin{array}{l}\text{Wood}\\ \text{Cardboard}\end{array} & \begin{bmatrix} 12 & 20 & 8 \\ 18 & 14 & 11 \end{bmatrix} = P \end{array}$$

a. Suppose Pearl wants to increase output by 25%. Write the matrix that represents the needed output. Round all decimals to the nearest whole number.

b. If your answer to Part a is the matrix kP, what is k?

13. Let $F = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{9} & \frac{1}{12} \end{bmatrix}$. Let $G = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$.

a. Solve $F = kG$ for k.

b. Solve $G = \ell F$ for ℓ.

14. Is $D = \begin{bmatrix} 20 & 12 \\ 50 & 40 \end{bmatrix}$ a scalar multiple of $E = \begin{bmatrix} \frac{1}{3} & \frac{1}{5} \\ \frac{5}{6} & \frac{3}{2} \end{bmatrix}$? If so, identify the scalar k. If not, explain why not.

In 15 and 16, solve for *a*, *b*, *c* and *d*.

15. $\begin{bmatrix} a & 18 \\ 54 & 35 \end{bmatrix} + \begin{bmatrix} 17 & b \\ c & d \end{bmatrix} = \begin{bmatrix} 21 & 81 \\ 25 & 30 \end{bmatrix}$

16. $2\begin{bmatrix} a & b \\ 14 & 4 \end{bmatrix} - 7\begin{bmatrix} 3 & -1 \\ c & -2.5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 4 & d \end{bmatrix}$

REVIEW

17. Write a matrix to represent the triangle with vertices at the origin, (0, 4), and (-1, 0). **(Lesson 4-1)**

18. According to the 2000 United States Census, in 2000 there were 7,229,068 foreign-born and 46,365,310 native-born people living in the Northeast; 3,509,937 foreign-born and 60,882,839 native-born people living in the Midwest; 8,608,441 foreign-born and 91,628,379 native-born people living in the South; and 11,760,443 foreign-born and 51,437,489 native-born people living in the West. Write a matrix to store this information in millions, rounding each population to the nearest million. **(Lesson 4-1)**

19. Graph $f(x) = -\lfloor x \rfloor$ and $g(x) = \lfloor -x \rfloor$ separately. Explain the difference in the two graphs. **(Lesson 3-9)**

20. A plumber charges a fixed fee to come to your house, plus an hourly fee for the time spent on the repair job. Suppose the plumber comes to your house, works for an hour, and charges \$120. The next week the plumber has to return to finish the job and spends one and a half hours working, and charges \$160. Write a possible equation that describes how much the plumber charges based on hours of work. Use it to figure out the fixed fee. **(Lesson 3-4)**

21. A rectangle has sides of length $4x$ and $0.5x$. **(Lesson 2-5)**

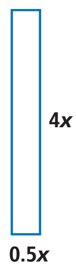
a. Write an equation for the area of the rectangle as a function of the lengths of its sides.

b. Graph this equation in an appropriate window.

c. What is the domain of the function you graphed?

d. What is the domain of the function if the context is ignored?

e. Graph the function again over the domain in Part d.



EXPLORATION

22. The matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ gives the endpoints of a line segment.

a. Find the slope of the line containing the segment and graph it.

b. Find the slope of the line through points $\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix}$ and graph it.

c. Compare the two matrices. Then generalize Parts a and b.

QY ANSWER

$\begin{bmatrix} 8273 & 7169 \end{bmatrix} - \begin{bmatrix} 8221 & 6813 \end{bmatrix} =$
$\begin{bmatrix} 8273 - 8221 & 7169 - 6813 \end{bmatrix} =$
$\begin{bmatrix} 52 & 356 \end{bmatrix}$ The subtraction checks.