Lesson

10-8 Using Matrices to Solve Systems

Vocabulary

inverse (of a matrix)

BIG IDEA By finding the inverse of a matrix, you can solve systems of linear equations.

The matrix method for solving systems follows a pattern like the one used to solve the equation $\frac{2}{7}x = 28$.

To solve this equation, you would multiply both sides of the equation by the number that makes the coefficient of x equal to 1. This is the multiplicative inverse of $\frac{2}{7}$, or $\frac{7}{2}$.

$$\frac{7}{2} \cdot \frac{2}{7}x = \frac{7}{2} \cdot 28$$
$$1 \cdot x = 98$$

When the coefficient is 1, the equation simplifies to become a statement of the solution, $x = 98$.

Mental Math

If $g(t) = -4t^2$, calculate

a. $g(10)$.

b. $g(5)$.

c. $\frac{g(10)}{g(5)}$.

d. $g(2)$.

The 2 × 2 Identity Matrix

Refer to the solution to the above equation. Working backwards, the key to the solution $x = 98$ is to have obtained $1 \cdot x = 98$ in the previous step. For a system of two linear equations, if (e, f) is the solution, then the solution can be written $\begin{cases} x = e \\ y = f \end{cases}$. What is the previous step?

Working backwards, this is the same as $\begin{cases} 1x + 0y = e \\ 0x + 1y = f \end{cases}$. The coefficient matrix of this system is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Recall from Question 10 in Lesson 10-7 that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2×2 identity matrix because when it multiplies a 2×2 or 2×1 matrix, it does not change that matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

Thus, you can solve a system with matrices if you can convert it into an equivalent system in which the coefficient matrix is the identity matrix. This is done by multiplying both sides of the original matrix equation by a new matrix, called the **inverse** of the coefficient matrix. You can use technology to help you find the inverse matrix.

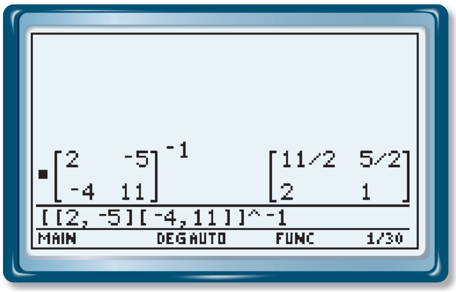
Example

Use matrices to solve the system $\begin{cases} 2x - 5y = 4 \\ -4x + 11y = -6 \end{cases}$.

Solution First, write the system in matrix form.

$$\begin{bmatrix} 2 & -5 \\ -4 & 11 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

Use technology to find the inverse of the coefficient matrix as shown on one particular calculator below.



So the inverse of $\begin{bmatrix} 2 & -5 \\ -4 & 11 \end{bmatrix}$ is $\begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix}$. Multiply each side of the matrix equation by $\begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix}$ on the left.

$$\underbrace{\begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -4 & 11 \end{bmatrix}} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -6 \end{bmatrix}}$$

This produces the identity matrix. Multiply these matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5.5 \cdot 4 + 2.5 \cdot -6 \\ 2 \cdot 4 + 1 \cdot -6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

The solution is $(x, y) = (7, 2)$.

Check Substitute $x = 7$ and $y = 2$ into each of the original equations.

Does $2 \cdot 7 - 5 \cdot 2 = 4$? Yes, $14 - 10 = 4$.

Does $-4 \cdot 7 + 11 \cdot 2 = -6$? Yes, $-28 + 22 = -6$.

Inverse 2 × 2 Matrices

In the previous example, we asserted that the inverse of $\begin{bmatrix} 2 & -5 \\ -4 & 11 \end{bmatrix}$ is $\begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix}$. While a calculator or computer may automatically give you the inverse, you still need to be able to check that what you are given is correct. This is done by doing the row-by-column multiplication,

$$\begin{bmatrix} 2 & -5 \\ -4 & 11 \end{bmatrix} \cdot \begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5.5 + -5 \cdot 2 & 2 \cdot 2.5 + -5 \cdot 1 \\ -4 \cdot 5.5 + 11 \cdot 2 & -4 \cdot 2.5 + 11 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

When two matrices are inverses, you can multiply them in either order and you will still get the identity matrix.

STOP **QY**

> **QY**
>
> Show that
> $\begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

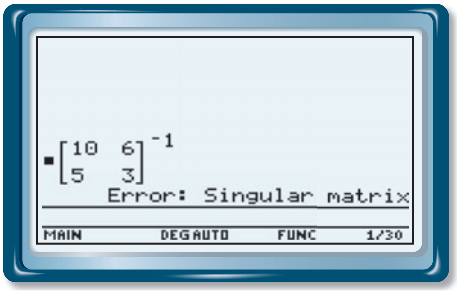
The inverse of the matrix A is denoted by the symbol A^{-1}. We write $\begin{bmatrix} 2 & -5 \\ -4 & 11 \end{bmatrix}^{-1} = \begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix}$. With powers of real numbers, $x \cdot x^{-1} = 1$, the multiplicative identity for real numbers. With matrices, $A \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the multiplicative identity for 2 × 2 matrices.

Summary of the Matrix Method

The matrix method of solving a system of linear equations is useful because it can be applied to systems with more than two variables in exactly the same way as it is applied to systems with two variables. You will work with these larger systems in later courses. The process is always the same.

$A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = B$	Write the system as the product of three matrices: coefficients • variables = constants.
$A^{-1} \cdot A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \cdot B$	Multiply (on the left) each side by the inverse of the coefficient matrix.
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \cdot B$	Because A and A^{-1} are inverses, their product is the identity matrix.
$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \cdot B$	

The left side gives the variables. The right side gives the solutions. *Caution:* Just as 0 has no multiplicative inverse, not all 2×2 matrices have multiplicative inverses. When a coefficient matrix does not have an inverse, then there is not a unique solution to the system. There may be infinitely many solutions, or there may be no solution.



Questions

COVERING THE IDEAS

In 1–3, consider the system $\begin{cases} 1x + 0y = 7 \\ 0x + 1y = -3 \end{cases}$.

1. Solve this system.
2. Write the coefficient matrix for the system.
3. Write the matrix form for the system.
4. Write the 2×2 identity matrix for multiplication.
5. Show that the inverse of $\begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$ is $\begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{5}{14} & \frac{1}{14} \end{bmatrix}$.
6. **Multiple Choice** Which of these matrices is the inverse of $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$?

   A $\begin{bmatrix} -3 & -4 \\ -5 & -7 \end{bmatrix}$ B $\begin{bmatrix} -2 & -4 \\ -5 & -6 \end{bmatrix}$ C $\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$ D $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{17} \end{bmatrix}$

In 7–9, use the given system.

a. Write the system in matrix form.
b. Use technology to find the inverse of the coefficient matrix.
c. Solve the system.

7. $\begin{cases} 3x + 5y = 27 \\ 2x + 3y = 17 \end{cases}$
8. $\begin{cases} 2x + 3y = 18 \\ 3x + 4y = 21 \end{cases}$
9. $\begin{cases} 2m - 6t = -6 \\ 7.5m - 15t = -37.5 \end{cases}$

APPLYING THE MATHEMATICS

10. a. Show that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ equals its multiplicative inverse.

    b. What real numbers equal their multiplicative inverses?

In 11 and 12, a system is given.

a. **Using inverse matrices, write the product of two matrices that will find the solution.**

b. **Give the solution.**

c. **Check the solution.**

11. $\begin{cases} 2.3x + y = -5.5 \\ 3.1x + 2.4y = -1.1 \end{cases}$

12. $\begin{cases} 0.5m + 1.5t = 10 \\ t + 14 = 0.5m \end{cases}$

13. Consider the three systems below.

i. $\begin{cases} x - 8y = -35 \\ 5x + 8y = 65 \end{cases}$ **ii.** $\begin{cases} y = x - 3 \\ 2x + 3y = 16 \end{cases}$ **iii.** $\begin{cases} 4x - 5y = 8 \\ 12x - 15y = 3 \end{cases}$

a. Choose the system which describes two parallel lines, and write its coefficient matrix.

b. What is your calculator's response when you try to find the inverse of the coefficient matrix of the system you chose in Part a?

14. Show that the matrix $\begin{bmatrix} 3 & 5 \\ 9 & 15 \end{bmatrix}$ has no multiplicative inverse by writing the matrix equation $\begin{bmatrix} 3 & 5 \\ 9 & 15 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as two systems of equations and showing that those systems have no solution.

REVIEW

In 15 and 16, multiply the given matrices. **(Lesson 10-7)**

15. $\begin{bmatrix} 5 & 6 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 0 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

17. Solve $\begin{bmatrix} -4 & 1 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ a \end{bmatrix} = \begin{bmatrix} -12 \\ 63 \end{bmatrix}$ for a. **(Lesson 10-7)**

In 18 and 19, determine whether the lines are parallel and nonintersecting, coincident, or intersecting in only one point. (Lesson 10-6)

18. $\begin{cases} y = 4x - 6 \\ 28x - 7y = -10 \end{cases}$

19. $\begin{cases} 8x + 6y = 10 \\ 4x - 3y = -5 \end{cases}$

20. A jar of change contains 64 coins consisting only of quarters and dimes. The total value of the coins in the jar is \$13.60. Let q = the number of quarters, and d = the number of dimes. **(Lesson 10-4)**

a. Write two equations that describe the information given above.

b. How many of each type of coin is in the jar?

Each dime has 118 reeds and each quarter has 119 reeds along its outer edge.

Source: U.S. Mint

21. ***Skill Sequence*** Solve each equation. (Lessons 9-5, 9-2, 8-6)

a. $x^2 = 144$

b. $x^2 + 44 = 144$

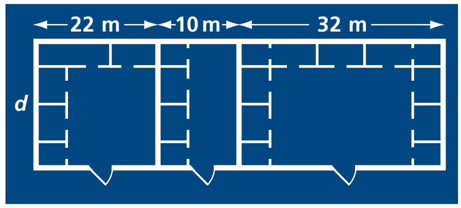
c. $4x^2 + 44 = 144$

d. $4x^2 + 44x + 265 = 144$

22. Simplify the expression $\sqrt{x} \cdot \sqrt{x} \cdot \sqrt{x^3}$. (Lessons 8-7, 8-6)

23. Suppose a bank offers a 4.60% annual yield on a 4-year CD. What would be the amount paid at the end of the 4 years to an investor who invests $1,800 in this CD? (Lesson 7-1)

24. Ms. Brodeur wants to lease about 2,000 square meters of floor space for a business. She noticed an advertisement in the newspaper regarding a set of 3 vacant stores. Their widths are given in the floor plan at the right. How deep must these stores be to meet Ms. Brodeur's required area? (Lesson 2-1)



EXPLORATION

25. A formula for the inverse of a 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be found by following these steps.

Step 1 Write $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & u \\ y & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as two systems of equations. Keep a, b, c, and d as the coefficients. One system will have the variables x and y and the other will have u and v.

Step 2 Solve for x, y, u, and v in terms of a, b, c, and d. Find a formula in this way and check it with at least two different matrices. You may want to use a CAS to find the formula.

QY ANSWER

$\begin{bmatrix} 5.5 & 2.5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -4 & 11 \end{bmatrix} =$

$\begin{bmatrix} 5.5 \cdot 2 + 2.5 \cdot -4 & 5.5 \cdot -5 + 2.5 \cdot 11 \\ 2 \cdot 2 + 1 \cdot -4 & 2 \cdot -5 + 1 \cdot 11 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$