Lesson 10-7 Matrices and Matrix Multiplication

Vocabulary

matrix (matrices)
elements
dimensions
matrix form
2 × 2 identity matrix

▶ **BIG IDEA** Rectangular arrays called matrices can sometimes be multiplied and represent systems of linear equations.

When you use a method like multiplication or addition to solve systems of linear equations, you do the same steps over and over. Once a linear system to be solved for x and y is in the form $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$, the processes for solving it are the same. The different solutions are caused by the numbers $a, b, c,$ and d that are the coefficients and the numbers e and f that are the constants.

A mathematical tool called a *matrix* allows you to separate those numbers from the overall structure of the problem. A **matrix** (the plural is **matrices**) is a rectangular array, such as $\begin{bmatrix} 3 & -4 \\ 15 & 0 \end{bmatrix}$.

The brackets [ ] identify the numbers that are in the matrix. The objects in the array are the **elements** of the matrix. The elements of the matrix $\begin{bmatrix} 3 & -4 \\ 15 & 0 \end{bmatrix}$ are 3, –4, 15, and 0. They are identified by the row and the column of the matrix they are in. The rows are counted from the top; the columns from the left. So –4 is the element in the 1st row and 2nd column.

The number of rows and the number of columns of a matrix are its **dimensions.** Because it has 2 rows and 2 columns, the matrix $\begin{bmatrix} 3 & -4 \\ 15 & 0 \end{bmatrix}$ is a 2 × 2 (read "2 by 2") matrix, while the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ is a 2 × 1 matrix because it has 2 rows and 1 column.

The linear system $\begin{cases} 2x + 6y = 2 \\ x + 4y = -5 \end{cases}$ is described by three matrices: the 2 × 2 *coefficient matrix* $\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$, the 2 × 1 *variable matrix* $\begin{bmatrix} x \\ y \end{bmatrix}$, and the 2 × 1 *constant matrix* $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

Mental Math

What is the probability that

a. a randomly-chosen one-digit number is odd?

b. a randomly-selected day in the year 2015 is in June?

c. a fair, 6-sided die shows a 3 or 5?

In *matrix form,* the system on the previous page is

$\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. In general, the **matrix form** of the system

$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$.

Example 1

Write $\begin{cases} 4x = 5y + 10 \\ 2x - 3y = 20 \end{cases}$ in matrix form.

Solution The first step is to rewrite the system with each equation in standard form: $\begin{cases} 4x - 5y = 10 \\ 2x - 3y = 20 \end{cases}$. Then form three matrices to describe the coefficients, variables, and constants in the standard-form system.

$$\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

coefficient matrix • variable matrix = constant matrix

Matrix Multiplication

Above, we have put a dot between the coefficient and variable matrices. This is because these matrices are multiplied. Matrices can be added, subtracted, and multiplied; but in this lesson you will learn only about matrix multiplication.

The matrix form of the system in Example 1 shows that the matrix $\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ is multiplied by $\begin{bmatrix} x \\ y \end{bmatrix}$. What does it mean to multiply these matrices? To multiply these matrices, we combine each row of the 2×2 matrix $\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ with the 2×1 matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ to form a product 2×1 matrix. Each element in the product matrix is the product of the first entries plus the product of the second entries. Multiplying the top row of $\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ by $\begin{bmatrix} x \\ y \end{bmatrix}$ gives $4x + -5y$. Multiplying the bottom row of $\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ by $\begin{bmatrix} x \\ y \end{bmatrix}$ gives $2x + -3y$.

So $\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - 5y \\ 2x - 3y \end{bmatrix}$. This is why we say that the system $\begin{cases} 4x - 5y = 10 \\ 2x - 3y = 20 \end{cases}$ is equivalent to the matrix equation $\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$. In the same way, two matrices can be multiplied when both matrices contain numbers.

Example 2

Perform the multiplication $\begin{bmatrix} 10 & 3 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 11 \end{bmatrix}$.

Solution The result will be a 2×1 matrix. So write down places for the elements of this matrix.

$$\begin{bmatrix} 10 & 3 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \begin{bmatrix} \underline{?} \\ \underline{?} \end{bmatrix}$$

Multiply the top row by the column to obtain the top element:

$10 \cdot 4 + 3 \cdot 11 = 73$. Multiply the bottom row by the column to obtain the bottom element: $-2 \cdot 4 + 5 \cdot 11 = 47$.

$$\begin{bmatrix} 10 & 3 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \begin{bmatrix} 10 \cdot 4 + 3 \cdot 11 \\ -2 \cdot 4 + 5 \cdot 11 \end{bmatrix} = \begin{bmatrix} 73 \\ 47 \end{bmatrix}$$

Multiplying 2 × 2 Matrices

Not all matrices can be multiplied. For a product AB of two matrices A and B to exist, each row of A must have the same number of elements as each column of B. This is so that row-by-column multiplication can be performed. The element in row i and column j of the product is the result of multiplying row i of A and the column j of B.

Example 3

Find the product $\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 6 \\ 30 & 5 \end{bmatrix}$.

Solution The product will be a 2×2 matrix. First write down the spaces for the elements of the product. The product will have the same number of rows as the first matrix and the same number of columns as the second matrix.

$$\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 6 \\ 30 & 5 \end{bmatrix} = \begin{bmatrix} \underline{?} & \underline{?} \\ \underline{?} & \underline{?} \end{bmatrix}$$

Pick an element of the product matrix.

For the element in the 1st row, 1st column of the product, multiply the 1st row of the left matrix by the 1st column of the right matrix.

$1 \cdot -4 + 2 \cdot 30 = 56$

$$\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 6 \\ 30 & 5 \end{bmatrix} = \begin{bmatrix} 56 & \underline{?} \\ \underline{?} & \underline{?} \end{bmatrix}$$

For the element in the 1st row, 2nd column of the product, multiply the 1st row of the left matrix by the 2nd column of the right matrix.

$1 \cdot 6 + 2 \cdot 5 = 16$

$$\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 6 \\ 30 & 5 \end{bmatrix} = \begin{bmatrix} 56 & 16 \\ \underline{?} & \underline{?} \end{bmatrix}$$

The other two elements are found in a similar manner.

$5 \cdot -4 + 3 \cdot 30 = 70$ and $5 \cdot 6 + 3 \cdot 5 = 45$

$$\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 6 \\ 30 & 5 \end{bmatrix} = \begin{bmatrix} 56 & 16 \\ 70 & 45 \end{bmatrix}$$

In matrix multiplication, the left and right matrices play different roles. So you should not expect that reversing the order of the matrices will give the same product. For the matrices of Example 3, $\begin{bmatrix} -4 & 6 \\ 30 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 26 & 10 \\ 55 & 75 \end{bmatrix}$. Matrix multiplication is not commutative.

Questions

COVERING THE IDEAS

1. Consider the matrix $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$.
   a. What are the dimensions of this matrix?
   b. Name the elements in the first row.
   c. Which element is in the 2nd row, 3rd column?

2. The matrix equation $\begin{bmatrix} -4 & 6 \\ 3 & -7 \end{bmatrix} \cdot \begin{bmatrix} d \\ g \end{bmatrix} = \begin{bmatrix} 18 \\ 54 \end{bmatrix}$ describes a system of equations. Write the system.

In 3 and 4, a system is given.
a. Write the coefficient matrix.
b. Write the constant matrix.

3. $\begin{cases} 5a - 2b = -4 \\ 3a + 4b = 34 \end{cases}$

4. $\begin{cases} 5x + 3(y + 1) = 85 \\ 2x = 7y \end{cases}$

5. What is the result when the row $[-4 \quad 6]$ is combined with the column $\begin{bmatrix} 0.25 \\ -0.50 \end{bmatrix}$ in a matrix multiplication?

In 6–8, multiply the two matrices.

6. $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

7. $\begin{bmatrix} 5 & -8 \\ 4 & 11 \end{bmatrix} \cdot \begin{bmatrix} 0.5 & 0 \\ -2 & 4 \end{bmatrix}$

8. $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ 5 & -6 \end{bmatrix}$

9. Give an example different from the one provided in this lesson to show that multiplication of 2×2 matrices is not commutative.

APPLYING THE MATHEMATICS

10. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the **2 × 2 identity matrix** for multiplication. To see why, calculate the products in Parts a and b.

    a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

    b. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

    c. **True or False** Matrix multiplication of a 2×2 matrix with the 2×2 identity matrix is commutative.

11. Solve $\begin{bmatrix} -9 & 2 \\ 0 & 15 \end{bmatrix} \cdot \begin{bmatrix} x \\ 5 \end{bmatrix} = \begin{bmatrix} 100 \\ 75 \end{bmatrix}$ for x.

12. Solve $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 4y \\ 2x + y \end{bmatrix}$ for $a, b, c,$ and d.

13. Create three different 2×2 matrices $M, N,$ and P.

    a. Calculate MN.

    b. Calculate $(MN)P$.

    c. Calculate NP.

    d. Calculate $M(NP)$.

    e. Do your answers to Parts b and d tell you that matrix multiplication is definitely not associative, or do they tell you that matrix multiplication might be associative?

14. When a matrix M is multiplied by itself, the product $M \cdot M$ is called M^2 for short. $M^2 \cdot M = M^3$, $M^3 \cdot M = M^4$, and so on. Let $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Show that M^4 is the identity matrix of Question 10.

REVIEW

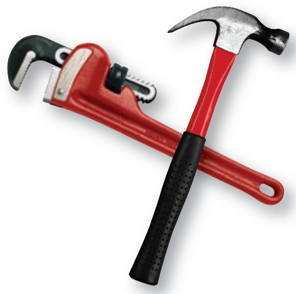
15. Without drawing any graphs, explain how you can tell whether the graphs of $17x + 20y = 84$ and $16x + 20y = 85$ are two intersecting lines, one line, or two nonintersecting parallel lines. (Lesson 10-6)

In 16 and 17, solve by using any method. (Lessons 10-6, 10-5, 10-4, 10-2, 10-1)

16. $\begin{cases} 3x + 2y = 40 \\ 9x + 6y = 120 \end{cases}$

17. $\begin{cases} y = 10 - 4x \\ y = 4x - 10 \end{cases}$

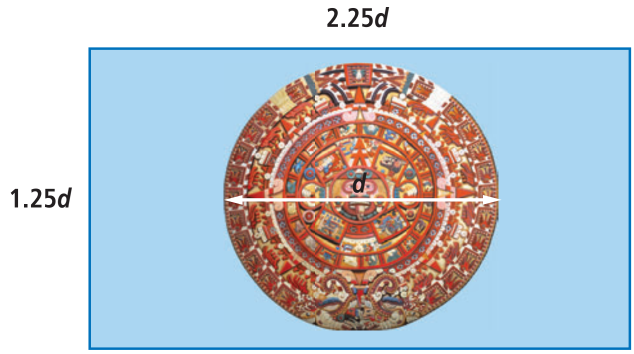
18. A hardware store placed two orders with a manufacturer. The first order was for 18 hammers and 14 wrenches, and totaled \$582. The second order was for 12 hammers and 10 wrenches, and totaled \$396. What is the cost of one hammer and of one wrench? (Lesson 10-5)



19. Consider the equations $2x + y = 4$, $x = 5$, and $y = 3$. The graph of these equations forms a triangle. (Lessons 10-1, 8-8, 8-6)
    a. Find the vertices of the triangle.
    b. Find the length of each side of the triangle.
    c. Find the area of the triangle.

20. ***Skill Sequence*** Find an equivalent expression without a fraction. (Lessons 8-4, 8-3)
    a. $\frac{x^5}{x}$ b. $\frac{x^5}{x^3}$ c. $\frac{x^5}{y^3}$

21. An Aztec calendar is being placed on a rectangular mat that is 1.25 times as high and 2.25 times as wide as the calendar. What percent of the mat is taken up by the calendar? (Lesson 5-7)



EXPLORATION

22. Rows and columns with 3 elements are multiplied as follows.

$$\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = [ad + be + cf]$$

3×3 matrices can be multiplied using the same row-by-column idea as is used with 2×2 matrices. The product MN of two 3×3 matrices M and N is a 3×3 matrix. The element in row i and column j of MN is the result of multiplying row i of M and the column j of N.

a. Use this idea to find MN when $M = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 5 & 2 \\ 0 & 3 & 10 \end{bmatrix}$ and

$N = \begin{bmatrix} 0 & 3 & -3 \\ 4 & 6 & 1 \\ -0.5 & 0 & 12 \end{bmatrix}$.

b. Show that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix for 3×3 matrix multiplication. (*Hint:* Calculate products as in Question 10.)