Lesson
10-5
Solving Systems by Multiplication

equivalent systems

multiplication method for solving a system

> ▶ **BIG IDEA** An effective first step in solving some systems is to multiply both sides of one of the equations by a carefully chosen number.

Recall that there are three common forms for equations of lines.

Mental Math

Classify the angle with the given measure as acute, right, or obtuse.

a. 134°

b. 84°

c. 0.23°

	Form	Example
Standard	$Ax + By = C$	$3x + 8y = 20$
Slope-Intercept	$y = mx + b$	$y = -2x + 1$
Point-Slope	$y - k = m(x - h)$	$y - 50 = \frac{3}{4}(x - 20)$

The substitution method described in Lessons 10-2 and 10-3 is convenient for solving systems in which one or both equations are in slope-intercept form. The addition method studied in Lesson 10-4 is convenient for solving systems in which both equations are in standard form and the coefficients of one variable are either equal or opposites. However, not all systems fall into one of these two categories.

Consider the following system.

$$\begin{cases} 3x - 4y = 7 \\ 6x - 5y = 20 \end{cases}$$

Adding or subtracting the two equations will not result in an equation with just one variable, because the x and the y terms are neither equal nor opposites. Substitution could be used, but it introduces fractions.

An easier method uses the Multiplication Property of Equality to create an *equivalent system* of equations. **Equivalent systems** are systems with exactly the same solutions. Notice that if you multiply both sides of the first equation by -2, the x terms of the resulting system have opposite coefficients.

Example 1

Solve the system $\begin{cases} 3x - 4y = 7 \\ 6x - 5y = 20 \end{cases}$.

Solution 1 Multiply both sides of the first equation by −2 and apply the Distributive Property.

$$\begin{cases} 3x - 4y = 7 \\ 6x - 5y = 20 \end{cases} \xrightarrow{\text{multiply by -2}} \begin{cases} -2(3x - 4y) = -2(7) \\ 6x - 5y = 20 \end{cases}$$

$$\begin{cases} -6x + 8y = -14 \\ 6x - 5y = 20 \end{cases}$$

$$3y = 6 \qquad \text{Add the equations.}$$
$$y = 2 \qquad \text{Solve for } y.$$

To find x, substitute 2 for y in one of the original equations.

$$3x - 4y = 7$$
$$3x - 4 \cdot 2 = 7$$
$$3x - 8 = 7$$
$$3x = 15$$
$$x = 5$$

So the solution is $(x, y) = (5, 2)$.

Solution 2 Multiply both sides of the second equation by $-\frac{1}{2}$. This also makes the coefficients of x opposites.

$$\begin{cases} 3x - 4y = 7 \\ 6x - 5y = 20 \end{cases} \xrightarrow{\text{multiply by } -\frac{1}{2}} \begin{cases} 3x - 4y = 7 \\ -\frac{1}{2}(6x - 5y) = -\frac{1}{2}(20) \end{cases}$$

$$\begin{cases} 3x - 4y = 7 \\ -3x + \frac{5}{2}y = -10 \end{cases}$$

$$-\frac{3}{2}y = -3 \qquad \text{Add.}$$
$$y = 2$$

Proceed as in Solution 1 to find x. *Again* $(x, y) = (5, 2)$.

Example 1 shows that the solution is the same no matter which equation is multiplied by a number. The goal is to obtain opposite coefficients for one of the variables in the two equations. Then the resulting equations can be added to eliminate that variable. This technique is sometimes called the **multiplication method for solving a system.**

🛑 **QY**

▶ **QY**

$$\begin{cases} 7x + 3y = 22.5 \\ 2x - 12y = 45 \end{cases}$$
$$\downarrow$$
$$\begin{cases} 28x + 12y = 90 \\ 2x - 12y = 45 \end{cases}$$

a. Explain what operation occurred to go from the first system to the second system.

b. Finish solving the system.

Sometimes it is necessary to multiply *each* equation by a different number before adding.

Example 2

Solve the system $\begin{cases} -3m + 2n = 6 \\ 4m + 5n = -31 \end{cases}$.

Solution The idea is to multiply by a number so that one variable in the resulting system has a pair of opposite coefficients. To make the coefficients of m opposites, multiply the first equation by 4 and the second equation by 3.

$$\begin{cases} -3m + 2n = 6 \\ 4m + 5n = -31 \end{cases} \xrightarrow[\text{multiply by 3}]{\text{multiply by 4}} \begin{cases} -12m + 8n = 24 \\ \underline{12m + 15n = -93} \end{cases}$$

$$\text{Now add.} \quad 23n = -69$$
$$n = -3$$

To find m, substitute -3 for n in either original equation. We use the first equation.

$$-3m + 2 \cdot (-3) = 6$$
$$-3m - 6 = 6$$
$$-3m = 12$$
$$m = -4$$

So $(m, n) = (-4, -3)$.

Check You should check your solution by substituting for m and n in each original equation.

Many situations naturally lead to linear equations in standard form. This results in a linear system that can be solved using the multiplication method.

Example 3

A marching band currently has 48 musicians and 18 people in the flag corps. The drum majors wish to form hexagons and squares like those diagrammed at the right. Are there enough members to create the formations with no people left over? If so, how many hexagons and how many squares can be made? If not, give a recommendation for the fewest people the drum majors would need to recruit and how many hexagons and how many squares could be made.

Hexagon
Flag bearer in center

Square
Two musicians in the center

Solution Consider the entire formation to include h hexagons and s squares. There are two conditions in the system: one for musicians and one for the flag corps.

There are $6\frac{\text{musicians}}{\text{hexagon}}$ and $2\frac{\text{musicians}}{\text{square}}$.

So $6h + 2s = 48$ musicians.

There are $1\frac{\text{flag bearer}}{\text{hexagon}}$ and $4\frac{\text{flag bearers}}{\text{square}}$.

So $h + 4s = 18$ flag bearers.

To find h, multiply the first equation by −2, and add the result to the second equation.

$$-12h + -4s = -96$$
$$\underline{h + 4s = 18}$$
$$-11h = -78$$
$$h = 7.\overline{09}$$

Because h is not a positive integer in this solution, these formations will not work with 48 musicians and 18 flag bearers.

−78 is not divisible by −11, but −77 is. By recruiting one additional member to the flag corps we get a number divisible by −11. This would create the following system.

$$-12h + -4s = -96$$
$$\underline{h + 4s = 19}$$
$$-11h = -77$$
$$h = 7$$

Now $h = 7$. Substituting for h in the second equation of this new system, you find that $s = 3$.

All 48 musicians and 19 flag bearers could be arranged into 7 hexagons and 3 squares, so the drum major needs to recruit 1 more flag bearer.

Check Making 7 hexagons would use 42 musicians and 7 flag bearers. Making 3 squares would use 6 musicians and 12 flag bearers. This setup uses exactly 48 musicians and 19 flag bearers.

Marching bands perform in competitions, at sporting events, and in parades.

When the equations in a system are not given in either standard or slope-intercept form, it is wise to rewrite the equations in one of these forms before proceeding. For example, to solve the system below, you could use one of three methods.

$$\begin{cases} n - 3 = \frac{3}{2}m \\ 4m + 5n = -31 \end{cases}$$

Method 1 Multiply the first equation by 2 to eliminate fractions.

$$\begin{cases} 2n - 6 = 3m \\ 4m + 5n = -31 \end{cases}$$

Add $-3m$ and 6 to both sides of the first equation. The result is the system of Example 2, which is in standard form.

Method 2 Add 3 to both sides of the first equation.

$$\begin{cases} n = \frac{3}{2}m + 3 \\ 4m + 5n = -31 \end{cases}$$

To finish solving this system you could use substitution by substituting n into the second equation.

Method 3 Use substitution on a CAS to solve the system.

Step 1 Use the SOLVE command to solve one of the equations for one of the variables. We choose the first equation and solve for n.

Step 2 Substitute this value for n into the second equation and solve for m. Most CAS will allow you to copy and paste so that you do not have to type expressions multiple times. The display shows $m = -4$.

Step 3 Then substitute -4 for m into the first equation to get $n = -3$.

Questions

COVERING THE IDEAS

1. Consider the system $\begin{cases} 5x + 3d = 9 \\ 2x + d = 26 \end{cases}$.

   a. **Fill in the Blanks** If the ___?___ equation is multiplied by ___?___, then adding the equations will eliminate ___?___.

   b. Solve the system.

2. A problem on a test was to solve the system $\begin{cases} -8n + m = -19 \\ 4n - 3m = -8 \end{cases}$.

Three students used three different methods to solve the system. Their first steps are shown.

Annisha's Method	Maxandra's Method	Victor's Method
$\begin{cases} -8n + m = -19 \\ 8n - 6m = -16 \end{cases}$	$\begin{cases} m = -19 + 8n \\ 4n - 3m = -8 \end{cases}$	$\begin{cases} -24n + 3m = -57 \\ 4n - 3m = -8 \end{cases}$

a. Which student(s) used substitution to solve the system?

b. Which variable will Annisha's method eliminate? Explain what she did to make an equivalent system.

c. Which variable will Victor's method eliminate? Explain what he did to make an equivalent system.

d. Pick one of the methods and finish solving the system.

3. Consider the system $\begin{cases} 7r - 3s = 9 \\ 2r + 5s = 26 \end{cases}$.

a. By what two numbers can you multiply the equations so that, if you add the results, you will eliminate r?

b. By what two numbers can you multiply the equations so that, if you add the results, you will eliminate s?

c. Use one of these methods to solve the system.

4. Consider the system $\begin{cases} 10t + u = 85 \\ 2t + 3u = 31 \end{cases}$.

a. Write an equivalent system that would eliminate t first.

b. Write an equivalent system that would eliminate u first.

c. Use one of the methods to solve the system.

5. A marching band has 60 musicians and 30 flag bearers. They wish to form pentagons and squares like those diagrammed at the right.
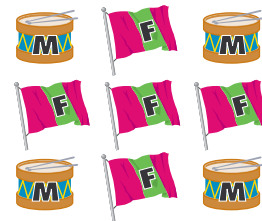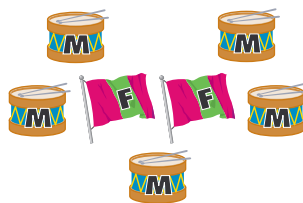
a. If the formation has 3 pentagons and 4 squares, how many musicians and flag bearers will be involved?

b. Is it possible to change the numbers of pentagons and squares so that every person will have a spot? If so, how many of each formation will be needed?

6. Solve the system $\begin{cases} n + 7 = \frac{1}{3}m \\ 7m - 3n = 57 \end{cases}$.

In 7–10, solve the system.

7. $\begin{cases} 24x + 15y = 20 \\ 4x + 3y = 5 \end{cases}$

8. $\begin{cases} 7a - 8b = 1 \\ 6a - 7b = 1 \end{cases}$

9. $\begin{cases} 9y + x = -8 \\ 2 = y - x \end{cases}$

10. $\begin{cases} 113.2 = 4x - 2y \\ 331.4 = 6x + 5y \end{cases}$

APPLYING THE MATHEMATICS

11. Solve the system by first rewriting each equation in standard form. $\begin{cases} 0.2x + 0.3(x + 4) = 0.16y \\ 0.04y - 0.07 = 0.08x \end{cases}$

12. Milo feels that the probability that he will be elected to the student council is $\frac{1}{10}$ of the probability that he will not be elected. What does Milo think is the probability that he will be elected? (Remember that the sum of the probabilities that he will be elected and not be elected is 1.)

13. A test has m multiple-choice (MC) questions and e extended-response (ER) questions. If the MC questions are worth 2 points each and the ER questions are worth 7 points each, the test will be worth a total of 95 points. If the MC questions are worth 3 points each and the ER questions are worth 8 points each, the test will be worth a total of 130 points. How many MC questions and how many ER questions are on the test?

Three students are participating in student council elections.

14. A security guard counted 82 vehicles in a parking lot. The only vehicles in the lot were cars and motorcycles. To double-check his count, the security guard counted 300 wheels. How many motorcycles and how many cars are in the parking lot?

15. Delise and Triston's class went on a field trip to a local farm. The farm raised cows and chickens. Delise counted 27 heads and Triston counted 76 legs. How many cows and how many chickens are on the farm?

    a. Answer this question by solving a system.

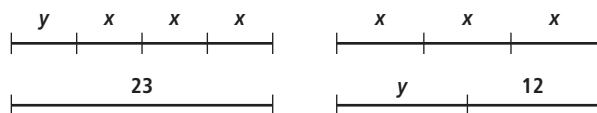   b. Write a few sentences explaining how to answer the question without using algebra.

REVIEW

In 16 and 17, solve the system using any method.
(Lessons 10-4, 10-3, 10-2, 10-1)

16. $\begin{cases} 6x + 2y = 26 \\ 4x + 2y = 8 \end{cases}$

17. $\begin{cases} y = \frac{2}{3}x - 4 \\ y = \frac{1}{4}x + 1 \end{cases}$

18. The two diagrams below illustrate a system of equations.
(**Lessons 10-4, 10-2**)

a. Write an equation for the diagram at the left.

b. Write an equation for the diagram at the right.

c. Solve the system for x and y.

d. Check your work.

19. Ashlyn has $600 and saves $30 each week. Janet has $1,500 and spends $30 each week. (**Lesson 10-2**)

a. How many weeks from now will they each have the same amount of money?

b. What will this amount be?

20. A 16-foot ladder leans against a house. If the base of the ladder is 6 feet from the base of the house, at what height does the top of the ladder touch the house? (**Lessons 9-1, 8-6**)

In 21 and 22, consider a garage with a roof pitch of $\frac{3}{12}$ at the right. The garage is to be 20 feet wide. (**Lessons 6-3, 5-9**)

21. What is the slope of $\overline{AB}$?

22. a. What is the height h of the roof?

b. Find the length r of one rafter.

23. **True or False** (**Lesson 5-6**)

a. Probabilities are numbers from 0 to 1.

b. A probability of 1 means that an event must occur.

c. A relative frequency of –1 cannot occur.
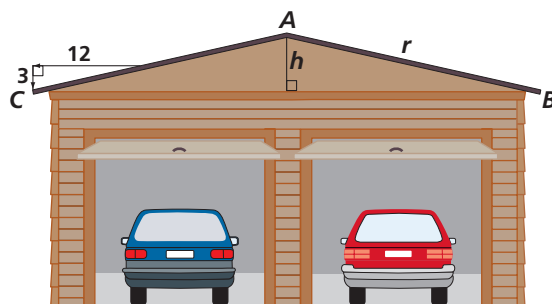
EXPLORATION

24. Create formations of a college band consisting of 110 musicians and a flag corps of 36 with no members left over.

QY ANSWERS

a. The first equation was multiplied by 4.

b. $(x, y) = (4.5, -3)$