Lesson 10-4 Solving Systems by Addition

Vocabulary

addition method for solving a system

BIG IDEA The sum of the left sides of two equations equals the sum of the right sides of those equations.

Mental Math

Find the perimeter of

a. a square with sides of length $6.2x$.

b. a regular octagon with sides of length $21ab$.

c. a regular pentagon with sides of length $4.5m + 1.5n$.

The numbers $\frac{1}{4}$ and 25% are equal even though they may not look equal; so are $\frac{17}{20}$ and 85%. If you add them, the sums are equal.

$$\frac{1}{4} = 25\% \qquad \frac{17}{20} = 85\%$$

So $\frac{1}{4} + \frac{17}{20} = 25\% + 85\%$.

Adding on each side, $\frac{22}{20} = 110\%$.

This is one example of the following generalization of the Addition Property of Equality.

Generalized Addition Property of Equality

For all numbers or expressions a, b, c, and d: If $a = b$ and $c = d$, then $a + c = b + d$.

The Generalized Addition Property of Equality can be used to solve some systems. Consider this situation: The sum of two numbers is 5,300. Their difference is 1,200. What are the numbers?

If x and y are the two numbers, with x the greater number, we can write the following system.

$$\begin{cases} x + y = 5{,}300 \\ x - y = 1{,}200 \end{cases}$$

Notice what happens when the left sides are added (combining like terms) and the right sides are added.

$$\begin{array}{r} x + y = 5{,}300 \\ \underline{+\ x - y = 1{,}200} \\ 2x + 0 = 6{,}500 \end{array}$$

Because y and $-y$ sum to 0, the sum of the equations is an equation with only one variable. Solve $2x = 6{,}500$ as usual.

$$x = 3{,}250$$

To find y, substitute 3,250 for x in one of the original equations. We choose $x + y = 5{,}300$.

$$x + y = 5{,}300$$
$$3{,}250 + y = 5{,}300$$
$$y = 2{,}050$$

The ordered pair (3,250, 2,050) checks in both equations:
$3{,}250 + 2{,}050 = 5{,}300$ and $3{,}250 - 2{,}050 = 1{,}200$.

So the solution to the system $\begin{cases} x + y = 5{,}300 \\ x - y = 1{,}200 \end{cases}$ is (3,250, 2,050).

Using the Generalized Addition Property of Equality to eliminate one variable from a system is sometimes called the **addition method for solving a system.** The addition method is an efficient way to solve systems when the coefficients of the same variable are opposites.

Example 1

A pilot flew a small plane 180 miles from North Platte, Nebraska, to Scottsbluff, Nebraska, in 1 hour against the wind. The pilot returned to North Platte in 48 minutes $\left(\frac{48}{60} = \frac{4}{5}\text{ hour}\right)$ with the wind at the plane's back. How fast was the plane flying (without wind)? What was the speed of the wind?

There are more than 8,100 airports in the United States used only by small planes. They have runways shorter than 3,000 feet.

Source: Aircraft Owners and Pilots Association

Solution Let A be the average speed of the airplane without wind and W be the speed of the wind, both in miles per hour. The total speed against the wind is then $A - W$, and the speed with the wind is $A + W$. There are two conditions given on these total speeds.

From North Platte to Scottsbluff the average speed of the plane was $\frac{180\text{ miles}}{1\text{ hour}} = 180\ \frac{\text{miles}}{\text{hour}}$.

This was against the wind, so $A - W = 180$.

From Scottsbluff to North Platte the average speed of the plane was $\frac{180\text{ miles}}{\frac{4}{5}\text{ hour}} = 225\ \frac{\text{miles}}{\text{hour}}$.

This was with the wind, so $A + W = 225$.

We have the system $\begin{cases} A - W = 180 \\ A + W = 225 \end{cases}$.

Now solve the system. Since the coefficients of W are opposites (1 and −1), add the equations.

$$A - W = 180$$
$$\underline{A + W = 225}$$
$$2A = 405 \quad \text{Add.}$$
$$A = 202.5 \quad \text{Divide by 2.}$$

Substitute 202.5 for A in either of the original equations. We choose the second equation.

$$202.5 + W = 225$$
$$W = 22.5$$

The average speed of the airplane was about 202.5 mph and the speed of the wind was 22.5 mph.

Check Refer to the original question. Against the wind, the plane flew at 202.5 − 22.5 or 180 mph, so it flew 180 miles in 1 hour. With the wind, the plane flew at 202.5 + 22.5 or 225 mph. At that rate, in 48 minutes the pilot flew $\frac{48}{60}$ hr • 225 $\frac{\text{mi}}{\text{hr}}$ = 180 miles, which checks with the given conditions.

Sometimes the coefficients of the same variable are equal. In this case, use the Multiplication Property of Equality to multiply both sides of one of the equations by –1. This changes all the numbers in that equation to their opposites. Then you can use the addition method to find solutions to the system.

Example 2

Solve $\begin{cases} 5x + 17y = 1 \\ 5x + 8y = -26 \end{cases}$.

Solution We rewrite the equations and number them to make it easy to refer to them later.

$$\begin{cases} 5x + 17y = 1 & \text{Equation \#1} \\ 5x + 8y = -26 & \text{Equation \#2} \end{cases}$$

Notice that the coefficients of x in the two equations are equal.

Multiply the second equation by −1. Call the resulting Equation #3.

$$-5x - 8y = 26 \qquad \text{Equation \#3}$$

Now use the addition method with the first and third equations.

$$\begin{array}{rl} 5x + 17y = 1 & \text{Equation \#1} \\ + \; -5x - 8y = 26 & \text{Equation \#3} \\ \hline 9y = 27 & \text{Equation \#1 + Equation \#3} \\ y = 3 & \end{array}$$

To find x, substitute 3 for y in one of the original equations.

$$5x + 17(3) = 1 \qquad \text{We use Equation \#1.}$$
$$5x + 51 = 1$$
$$5x = -50$$
$$x = -10$$

So $(x, y) = (-10, 3)$.

(continued on next page)

Check Substitute in both equations.

Equation #1 Does 5 • -10 + 17 • 3 = 1? Yes.

Equation #2 Does 5 • -10 + 8 • 3 = -26? Yes.

GUIDED

Example 3

A resort hotel offers two weekend specials.

Plan A: 3 nights with 6 meals for $564

Plan B: 3 nights with 2 meals for $488

At these rates, what is the cost of one night's lodging and what is the average cost per meal? (Assume there is no discount for 6 meals.)

Solution Let N = price of one night's lodging.

Let M = average price of one meal.

Write an equation to describe each weekend special.

From Plan A: $3N + 6M = 564$ Equation #1

From Plan B: __?__ Equation #2

Notice the coefficients of N are the same, so multiply Equation #2 by −1.

__?__ Equation #3

__?__ Add Equations #1 and #3.

Does your last equation have only one variable? If so, solve this equation. If not, ask someone for help.

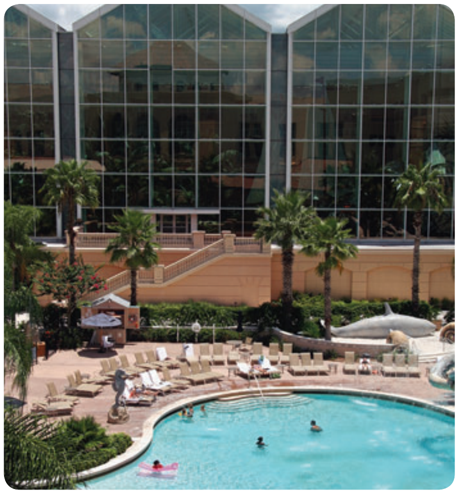
M = __?__

Substitute this value of M in either equation, and solve for N.

(N, M) = (__?__, __?__)

What is the price of one night's lodging? __?__

What is the average cost of a meal? __?__



The average hotel room rate in the United States in 2006 was $96.42 per night.

Source: Smith Travel Research

Questions

COVERING THE IDEAS

1. a. When is adding equations an appropriate method for solving systems?
   b. What is the goal in adding equations to solve systems?
2. Which property allows you to add to both sides of two equations to get a new equation?

In 3 and 4, a system is given.

a. Solve the system.

b. Check your solution.

3. $\begin{cases} 3x + 9y = 75 \\ -3x - y = 15 \end{cases}$

4. $\begin{cases} a + b = -22 \\ a - b = 4 \end{cases}$

5. The sum of two numbers is 1,776 and their difference is 1,492. What are the numbers?

6. Find two numbers whose sum is 20 and whose difference is 20.

7. When is it useful to multiply an equation by –1 as a first step in solving a system?

8. Airlines schedule about 5.5 hours of flying time for an A320 Airbus to fly from Dulles International Airport near Washington, D.C., to Los Angeles International Airport. Airlines schedule about 4.5 hours of flying time for the reverse direction. The distance between these airports is about 2,300 miles. They allow about 0.4 hour for takeoff and landing.

a. From this information, estimate (to the nearest 5 mph) the average wind speed the airlines assume in making their schedule.

b. What average airplane speed (to the nearest 5 mph) do the airlines assume in making their schedule?

In 9 and 10, solve the system.

9. $\begin{cases} 14x - 5y = 9 \\ 17x - 5y = 27 \end{cases}$

10. $\begin{cases} 17m + 7n = 8 \\ 17m + 5n = 13 \end{cases}$

11. $(N, M) = (150, 19)$ is the solution to the system of equations $\begin{cases} 3N + 6M = 564 \\ 3N + 2M = 488 \end{cases}$ in Example 3. Check this solution.

12. A hotel offers the following specials. Plan A includes a two-night stay and one meal for \$199. Plan B includes a 2-night stay and 4 meals for \$247. What price is this per night and per meal?

APPLYING THE MATHEMATICS

In 13 and 14, solve the system.

13. $\begin{cases} 2x - 6y = 34 \\ x = 2 - 6y \end{cases}$

14. $\begin{cases} \frac{1}{4}z + \frac{3}{4}w = \frac{1}{2} \\ \frac{7}{4}w + \frac{1}{4}z = \frac{3}{8} \end{cases}$

15. As you know, $\frac{3}{5} = 60\%$ and $\frac{3}{8} = 37.5\%$.

a. Is it true that $\frac{3}{5} - \frac{3}{8} = 60\% - 37.5\%$? Justify your answer.

b. Is it true that $\frac{3}{5} \cdot \frac{3}{8} = 60\% \cdot 37.5\%$? Justify your answer.

16. In 2006, the tallest person playing professional basketball in the Women's National Basketball Association (WNBA) was Margo Dydek. The shortest person was Debbie Black. When they stood next to each other, Margo was 23 in. taller. If one stood on the other's head, they would have stood 12 ft 5 in. tall. How tall is each player?

Margo Dydek

Debbie Black

REVIEW

In 17 and 18, solve by using any method. **(Lessons 10-3, 10-2, 10-1)**

17. $\begin{cases} y = 2x - 3 \\ y = -8x + 6 \end{cases}$

18. $\begin{cases} A = -5n \\ B = 6n \\ 4A + B = 39 \end{cases}$

19. **a.** Solve $x^2 + 3x - 28 = 0$.

b. Find the x-intercepts of the graph of $y = x^2 + 3x - 28$. **(Lesson 9-5)**

20. The formula $d = 0.04s^2 + 1.5s$ gives the approximate distance d in feet needed to stop a particular car traveling on dry pavement at a speed of s miles per hour. How much farther will this car travel before stopping if it is traveling at 65 mph instead of 50 mph? **(Lesson 9-3)**

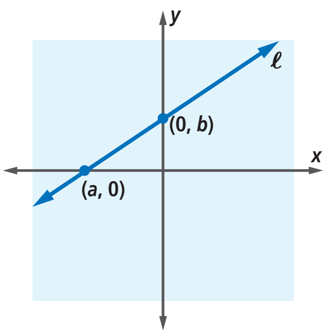
21. Let $f(x) = \sqrt{2x - 9}$. **(Lessons 8-6, 7-6, 7-5)**

a. What is the domain of f?

b. What is the range of f?

22. Simplify $x^{-1} + x - \frac{1}{x}$. **(Lessons 8-4, 8-3)**

23. Find the slope of line ℓ pictured below. **(Lesson 6-2)**



24. In 2005, the total revenues of a cell phone company increased 5.5% from the previous year to $36.84 billion. What were the company's revenues in 2004? **(Lesson 4-1)**

25. Solve $38(212 - x) = 0$ in your head. **(Lesson 3-4)**

EXPLORATION

26. Subtracting equations is part of a process that can be used to find simple fractions for repeating decimals. For example, to find a fraction for $0.\overline{72} = 0.7272727272...$, first let $d = 0.\overline{72}$. Then multiply both sides of the equation by an appropriate power of 10. Here we multiply by 10^2 because $0.\overline{72}$ has a two-digit block that repeats.

$100d = 72.\overline{72}$ Equation #1

$d = 0.\overline{72}$ Equation #2

Subtract the second equation from the first.

$99d = 72$ Equation #1 − Equation #2

Solve for d and simplify the fraction.

$$d = \frac{72}{99} \text{ or } d = \frac{8}{11}$$

A calculator shows that $\frac{8}{11} = 0.7272727272....$

a. Use the above process to find a simple fraction equal to $0.\overline{15}$.

b. Modify the process to find a simple fraction equal to $0.\overline{902}$.

c. Find a simple fraction equal to $0.\overline{123456}$.