Lesson 11-7 Permutations

Vocabulary

permutation

$n!$, n factorial

circular permutation

BIG IDEA From a set of n symbols, the number of permutations of length r without replacement is given by the product of the polynomials $\underbrace{n(n-1)(n-2)\ldots}_{r \text{ factors}}$.

In Lesson 8-1, you saw the following problem: How many 3-letter acronyms (like JFK, IBM, BMI, or TNT) are there in English? The answer is 26^3 because the first letter can be any one of the 26 letters of the alphabet, and so can the second letter, and so can the third letter. Think of the spaces to be filled and use the Multiplication Counting Principle: $\underline{26} \cdot \underline{26} \cdot \underline{26} = 17{,}576$.

Now suppose that the letters in the acronym have to be different. Then TNT and other acronyms with duplicate letters are not allowed. The first letter can still be any one of the 26 letters of the alphabet, but the second letter can only be one of the 25 letters remaining, and the third letter can only be one of the 24 letters remaining. So the total number of 3-letter acronyms in English with different letters is $\underline{26} \cdot \underline{25} \cdot \underline{24} = 15{,}600$.

The first situation is called an *arrangement with replacement* because a letter can be used more than once. The second situation, where a letter cannot be used more than once, is called a **permutation.** So, a permutation is an arrangement without replacement. The above situation shows that, with 26 letters, there are 17,576 arrangements with replacement of length 3, and 15,600 permutations of length 3.

Suppose there were only three letters, A, B, and C. Notice the difference between the two types of arrangements.

Length 2 with replacement	AA, AB, AC, BA, BB, BC, CA, CB, CC
Length 2 without replacement	AB, AC, BA, BC, CA, CB
Length 3 with replacement	AAA, AAB, AAC, ABA, ABB, ABC, ACA, ACB, ACC, 9 starting with B, and 9 more starting with C, a total of 3^3 or 27 arrangements
Length 3 without replacement	ABC, ACB, BAC, BCA, CAB, CBA
Length 4 with replacement	AAAA, AAAB, AAAC, ..., AABA, AABB, AABC, ... and so on, a total of 3^4 or 81 arrangements
Length 4 without replacement	None! (Do you see why?)

Mental Math

Suppose n is an integer. Determine if the following statements are *always*, *sometimes but not always*, or *never true*.

a. If $x > 0$, $x^n > 0$.

b. If $x < 0$, $x^n > 0$.

c. If $x < 0$, $2x^n < 0$.

Example 1

Dori saw a license plate with the numbers 15973. She noticed that all the digits were different odd digits. She wondered if this was unusual.

a. How many 5-digit numbers are there with only odd digits?

b. How many 5-digit numbers are there with only odd digits, all of them different?

Solution

a. This is a situation of arrangements with replacement. Each digit could be any of 5 numbers. Think of spaces to be filled.

$$\underline{5} \cdot \underline{5} \cdot \underline{5} \cdot \underline{5} \cdot \underline{5} = 3{,}125$$

b. This is a situation of permutations. The first digit can be any one of the 5 odd numbers. But then the second digit must be different, so it can only be one of the 4 odd numbers that remain. The third digit can only be one of the 3 odd numbers that remain after the first two have been chosen. The 4th digit can only be one of the 2 that remain. The fifth digit can only be the remaining digit. Filling spaces, you can picture this as $\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 120$.

There is quite a difference between the two kinds of arrangements! So it is not so unusual that Dori saw a license plate with all odd digits, but rather unusual that all the digits would be different.

GUIDED

Example 2

Three of the 11 members of a jazz band will each perform a solo at a concert. In how many orders can three people be picked from the band to perform a solo?

Solution This is an arrangement __?__ replacement situation.
(with/without)

Any of __?__ people could perform the first solo.

After that soloist, any of __?__ people could perform the second solo.

Then any of __?__ people could perform the third solo.

So there are __?__ • __?__ • __?__ or __?__ possible orders.

Jazz music developed in the last part of the 19th century in New Orleans.

Source: *Columbia Encyclopedia*

A Connection with Polynomials

Suppose you begin with n letters. *Without* replacement, there are $\underline{n} \cdot \underline{(n-1)}$ acronyms of length 2. Using the Distributive Property, you can see that $n(n-1) = n^2 - n$, which is a polynomial of degree 2. There are $\underline{n} \cdot \underline{(n-1)} \cdot \underline{(n-2)}$ acronyms of length 3.

Using the Extended Distributive Property, this product equals $n^3 - 3n^2 + 2n$, a polynomial of degree 3. Example 1b asked for permutations of length 5. Since $n = 5$ in Example 1b, the multiplication was $\underline{5} \cdot (\underline{5-1}) \cdot (\underline{5-2}) \cdot (\underline{5-3}) \cdot (\underline{5-4})$. In general, there are $\underline{n} \cdot (\underline{n-1}) \cdot (\underline{n-2}) \cdot (\underline{n-3}) \cdot (\underline{n-4})$ such permutations.

This product equals $n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$, a 5th degree polynomial. In general, the number of permutations of length n can be calculated by evaluating a polynomial of degree n.

Permutations Using All the Items

With five different items, there cannot be permutations of length 6 because there are only 5 different items. However, the situation of permutations using all the items is quite common, as Example 3 shows.

Example 3

You have 13 books to put on a shelf. In how many ways can they be arranged?

Solution This is a permutation problem. Any one of the 13 books can be farthest left. Once it has been chosen, there are 12 choices for the book to its right. Then there are 11 choices for the book to the right of the first two; and so on. The total number of permutations of the books is $\underline{13} \cdot \underline{12} \cdot \underline{11} \cdot \underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$, or 6,227,020,800.

The answers to Examples 1b and 3 each are the product of the integers from 1 to n. This product is denoted as $n!$ and called ***n* factorial.** Using factorial notation, the answer to Example 1b is 5! and the answer to Example 3 is 13!

$$n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Specifically, $1! = 1$.

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$, and so on.

Notice how $n!$ gets large quite quickly as n grows.

STOP QY

QY

a. A baseball manager is setting the batting order for the 9 starting players. How many different batting orders are possible?

b. What is 8!?

Factorial notation is very convenient for describing numbers of permutations. You saw at the beginning of this lesson the product of three numbers 26 • 25 • 24. This product can be written as the quotient of two factorials.

$$\begin{aligned} 26 \cdot 25 \cdot 24 &= 26 \cdot 25 \cdot 24 \cdot \frac{23 \cdot 22 \cdot 21 \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{23 \cdot 22 \cdot 21 \cdot \ldots \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{23 \cdot 22 \cdot 21 \cdot \ldots \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{26!}{23!} \end{aligned}$$

So, with a factorial key on a calculator, you can calculate any number of permutations just by dividing two factorials.

Questions

COVERING THE IDEAS

1. a. Identify two of the permutations of length 4 from the six letters A, B, C, D, E, and F.
   b. How many permutations are there of length 4 from the six letters A, B, C, D, E, and F?
2. You have 8 of your favorite pictures that you would like to hang on a wall, but there is room for only 3 of the pictures. How many different permutations of 3 pictures are possible from your 8 favorites, assuming you arrange the pictures in a straight line so that their order matters?
3. How many different permutations of length 2 are there from n objects when $n \geq 2$?
4. How many different permutations of length 7 are there from n objects when $n \geq 7$?
5. a. Identify two of the permutations of length 5 from the five letters V, W, X, Y, and Z.
   b. How many permutations of length 5 are there from these five letters?
6. A volleyball coach is deciding the serving order for the six starting players. How many different starting orders are possible?
7. In how many different orders can n objects be arranged on a shelf?
8. Give the values of 7!, 8!, 9!, and 10!.

In 9–12, write as a single number in base 10 or as a simple fraction in lowest terms.

9. $\frac{8!}{6!}$
10. $\frac{100!}{99!}$
11. $\frac{15!}{17!}$
12. $\frac{2!}{6!}$

The forearm pass, or "dig" is a type of shot used when receiving a serve or playing a hard, low hit ball.

APPLYING THE MATHEMATICS

13. Consider the following pattern.

$2! = 2 \cdot 1!$ $\qquad$ $3! = 3 \cdot 2!$ $\qquad$ $4! = 4 \cdot 3!$

Are all three instances above true? If so, describe the general pattern using one variable. If not, correct any instances that are false.

14. Three hundred people enter a raffle. The first prize is a computer and the second prize is a cell phone.

a. How many different ordered pairs of people are eligible to win these prizes?

b. Suppose you and your best friend are among the 300. If the winning tickets are chosen at random, what is the probability that you will win the computer and your best friend will win the cell phone?

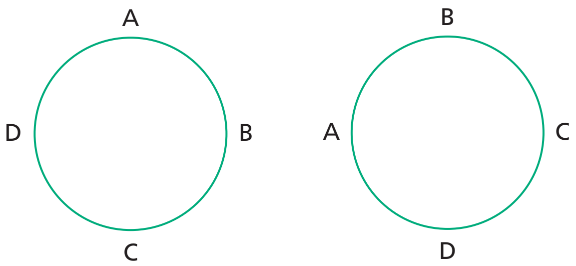
A raffle is used in many organizations looking to raise funds.

15. a. How many license plate numbers with 4 digits have all different odd digits, with the first digit being 3?

b. Write down all these license plate numbers.

16. How many license plate numbers with 6 digits have all different odd digits?

17. A **circular permutation** is an ordering of objects around a circle. Some permutations that are different along a line are considered the same around a circle. For example, the two arrangements of A, B, C, and D pictured here are considered to be the same.



Write all the circular permutations of 4 objects around a circle.

REVIEW

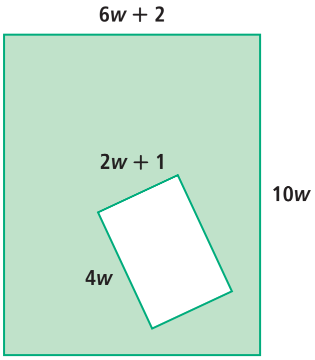
18. Expand and simplify $(-a + 6b)(-a - 6b)$. **(Lesson 11-6)**

19. How much smaller is the area of a circle with radius $(r - 4)$ than that of a circle with radius r (assume $r > 4$)? **(Lesson 11-5)**

In 20 and 21, simplify the expression. (Lessons 11-4, 8-3)

20. $\dfrac{25n^2 - 21n}{n}$ $\qquad$ **21.** $\dfrac{8x^3 + 16x^2 + 24x^6}{4x^2}$

22. A rectangle with dimensions $4w$ and $2w + 1$ is contained in a rectangle with dimensions $10w$ and $6w + 2$, as shown in the diagram at the right. **(Lesson 11-3)**



a. Write an expression for the area of the larger rectangle.

b. Write an expression for the area of the smaller rectangle.

c. Write a polynomial in standard form for the area of the shaded region.

23. Consider the polynomial $3x^3 - 4x^2 + x + 6$. Give an example of a polynomial of degree 3 that when added to this will give a polynomial of degree 1. **(Lesson 11-2)**

24. The spreadsheet below shows an investment of \$1,200 increasing in two different ways. **(Lesson 7-7)**

◇	A	B	C
1	Years From Now	Exponential Growth	Constant Increase
2	0	1200	1200
3	1	1260	1275
4	2	1323	1350
5	3		
6	4		
7	5		
8	6		
9	7		

a. What formula should be entered in cell B5? C5?

b. Describe the difference in investments after 7 years.

c. Which investment will be worth more in 10 years? Justify your answer.

EXPLORATION

25. Here are polynomial expressions for the number of permutations of n symbols with various lengths:

Length 2: $n(n - 1) = n^2 - n$

Length 3: $n(n - 1)(n - 2) = n^3 - 3n^2 + 2n$

Length 4: $n(n - 1)(n - 2)(n - 3) = n^4 - 6n^3 + 11n^2 - 6n$

Length 5: $n(n - 1)(n - 2)(n - 3)(n - 4) = n^5 - 10n^4 + 35n^3 - 50n^2 + 24n$

a. Use a CAS to find a polynomial expression for the number of permutations of n symbols with length 6.

b. Identify some patterns in these polynomials that enable you to predict some features of the polynomial for the number of permutations of n symbols with length 7.

QY ANSWERS

1. 362,880

2. 40,320