

Lesson

11-6

Special Binomial Products

Vocabulary

perfect square trinomials
difference of squares

► BIG IDEA The square of a binomial $a + b$ is the expression $(a + b)^2$ and can be found by multiplying $a + b$ by $a + b$ as you would multiply any polynomials.

Can you compute $46 \cdot 54$ in your head? How about 103^2 ? Studying products of special binomials can help you find the answers quickly without a calculator. Two such products are used so frequently that they are given their own names: Perfect Squares and the Difference of Two Squares.

Perfect Squares: The Square of a Sum

Just as numbers and variables can be squared, so can algebraic expressions. Given any two numbers a and b , you can expand $(a + b)^2$ or $(a - b)^2$. These are read “ a plus b , quantity squared” and “ a minus b , quantity squared.”

How can you expand $(a + b)^2$? One way is to write the power as repeated multiplication.

$$(a + b)^2 = (a + b)(a + b)$$

Next, use the Distributive Property.

$$= a(a + b) + b(a + b)$$

Then apply the Distributive Property again to the first and second products.

$$= (a^2 + ab) + (ba + b^2)$$

And finally combine like terms (because $ab = ba$).

$$= a^2 + 2ab + b^2$$

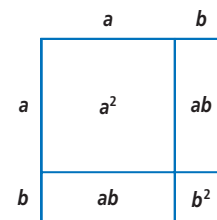
The square of a sum of two terms is the sum of the squares of the terms plus twice their product.

Geometrically, $(a + b)^2$ can be thought of as the area of a square with sides of length $a + b$. As the figure shows, its area is $a^2 + 2ab + b^2$.

Mental Math

A circle has diameter 10 centimeters. Estimate

- its circumference.
- its area.



STOP QY

► QY

Expand $(x + 8)^2$.

Example 1Calculate 103^2 .

Solution 1 Write 103 as the sum of two numbers whose squares you can calculate in your head. $103 = 100 + 3$, so $103^2 = (100 + 3)^2$.

Then use the special binomial product rule for the square of a sum.
 $(100 + 3)^2 = 100^2 + 2 \cdot 3 \cdot 100 + 3^2 = 10,000 + 600 + 9$
 $= 10,609$

Solution 2 Write the square as a multiplication and expand.

$$\begin{aligned} 103^2 &= (100 + 3)(100 + 3) \\ &= 100 \cdot 100 + 100 \cdot 3 + 3 \cdot 100 + 3 \cdot 3 \\ &= 10,000 + 300 + 300 + 9 = 10,609 \end{aligned}$$

With practice, either of the solutions to Example 1 can be done in your head.

GUIDED**Example 2**

The area of a square with side $7c + 5$ is $(7c + 5)^2$. Expand this binomial.

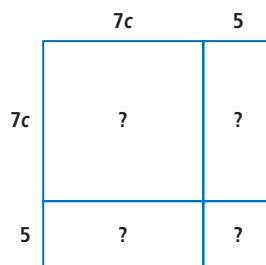
Solution 1 Use the rule for the square of a binomial.

$$\begin{aligned} (7c + 5)^2 &= (7c)^2 + \underline{\quad ? \quad} + \underline{\quad ? \quad} \\ &= \underline{\quad ? \quad} c^2 + \underline{\quad ? \quad} c + \underline{\quad ? \quad} \end{aligned}$$

Solution 2 Rewrite the square as a multiplication and expand using the Distributive Property.

$$\begin{aligned} (7c + 5)^2 &= (7c + 5)(7c + 5) \\ &= \underline{\quad ? \quad} (7c + 5) + \underline{\quad ? \quad} (7c + 5) \\ &= \underline{\quad ? \quad} \cdot 7c + \underline{\quad ? \quad} \cdot 5 + \underline{\quad ? \quad} \cdot 7c + \underline{\quad ? \quad} \cdot 5 \\ &= \underline{\quad ? \quad} c^2 + \underline{\quad ? \quad} c + \underline{\quad ? \quad} \end{aligned}$$

Solution 3 Draw a square with side $7c + 5$. Subdivide it into smaller rectangles and find the sum of their areas.



Check Test a special case. Let $c = 3$. Then $7c + 5 = \underline{\quad ? \quad}$ and $(7c + 5)^2 = \underline{\quad ? \quad}$.

Also $\underline{\quad ? \quad} c^2 + \underline{\quad ? \quad} c + \underline{\quad ? \quad} = \underline{\quad ? \quad} \cdot 9 + \underline{\quad ? \quad} \cdot 3 + \underline{\quad ? \quad} = \underline{\quad ? \quad}$.
 It checks.

Perfect Squares: The Square of a Difference

To square the difference $(a - b)$, think of $a - b$ as $a + -b$. Then apply the rule for the perfect square of a sum.

$$\begin{aligned}(a - b)^2 &= (a + -b)^2 \\ &= a^2 + 2a(-b) + (-b)^2 \\ &= a^2 - 2ab + b^2\end{aligned}$$

The square of a difference of two terms is the sum of the squares of the terms minus twice their product.

Squaring a binomial always results in a trinomial. Trinomials of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ are called **perfect square trinomials** because each is the result of squaring a binomial.

Perfect Squares of Binomials

For all real numbers a and b , $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$.

Activity 1

Complete the table.

$(a + b)^2$	$a^2 + 2ab + b^2$	$(a - b)^2$	$a^2 - 2ab + b^2$
$(x + 1)^2$?	$(x - 1)^2$?
$(x + 2)^2$?	$(x - 2)^2$?
$(x + 3)^2$?	$(x - 3)^2$?
$(x + 4)^2$?	$(x - 4)^2$?
$(x + 15)^2$?	$(x - 15)^2$?
$(x + n)^2$?	$(x - n)^2$?

The Difference of Two Squares

Another special binomial product is the sum of two numbers times their difference. Let x and y be any two numbers. What is $(x + y)(x - y)$?

$$\begin{aligned}(x + y)(x - y) &= x(x - y) + y(x - y) && \text{Distributive Property} \\ &= x^2 - xy + yx - y^2 && -xy \text{ and } yx \text{ are opposites.} \\ &= x^2 - y^2\end{aligned}$$

The product of the sum and difference of two numbers is the **difference of squares** of the two numbers.

Difference of Two Squares

For all real numbers x and y , $(x + y)(x - y) = x^2 - y^2$.

Activity 2

Complete the table at the right.

The difference of two squares can be used to multiply two numbers that are equidistant from a number whose square you know.

$(a + b)(a - b)$	$a^2 - b^2$
$(x + 1)(x - 1)$?
$(x + 2)(x - 2)$?
$(x + 3)(x - 3)$?
$(x + 4)(x - 4)$?
$(x + 15)(x - 15)$?
$(x + n)(x - n)$?

Example 3

Compute $46 \cdot 54$ in your head.

Solution 46 and 54 are the same distance from 50. So think of $46 \cdot 54$ as $(50 - 4)(50 + 4)$. This is the product of the sum and difference of the same numbers, so the product is the difference of the squares of the numbers.

$$(x - y)(x + y) = x^2 - y^2$$

$$(50 - 4)(50 + 4) = 50^2 - 4^2 = 2,500 - 16 = 2,484$$

Example 4

Expand $(8x^5 + 3)(8x^5 - 3)$.

Solution This is the sum of and difference of the same numbers, so the product is the difference of squares of the numbers.

$$(8x^5 + 3)(8x^5 - 3) = (8x^5)^2 - 3^2 = 64x^{10} - 9$$

Check Let $x = 2$.

$$\begin{aligned} (8x^5 + 3)(8x^5 - 3) &= (8 \cdot 2^5 + 3)(8 \cdot 2^5 - 3) \\ &= (8 \cdot 32 + 3)(8 \cdot 32 - 3) \\ &= 259 \cdot 253 = 65,527 \end{aligned}$$

$$64x^{10} - 9 = 64 \cdot 2^{10} - 9 = 64 \cdot 1,024 - 9 = 65,527, \text{ so it checks.}$$

Questions

COVERING THE IDEAS

In 1–3, expand and simplify the expression.

- $(g + h)^2$
- $(g - h)^2$
- $(g + h)(g - h)$

4. What is a *perfect square trinomial*?
5. Give an example of a perfect square trinomial.

In 6 and 7, a square is described.

- a. Draw a picture to describe the situation.
 - b. Write the area of the square as the square of a binomial.
 - c. Write the area as a perfect square trinomial.
6. A square with sides of length $2n + 1$.
 7. A square with sides of length $5p + 11$.
 8. Verify that $(a - b)^2 = a^2 - 2ab + b^2$ by substituting numbers for a and b .

In 9–16, expand and simplify the expression.

9. $(x - 5)^2$
10. $(3 + n)(3 - n)$
11. $(n^2 + 4)(n^2 - 4)$
12. $(13s + 11)^2$
13. $(9 - 2x)^2$
14. $\left(10 + \frac{1}{2}t\right)^2$
15. $(3x + yz)(3x - yz)$
16. $(2a + 5b)(-5b + 2a)$
17. Compute in your head. Then write down how you did each computation.
 - a. 30^2
 - b. $29 \cdot 31$
 - c. $28 \cdot 32$
 - d. $27 \cdot 33$

In 18–20, compute in your head. Then write down how you did each computation.

18. $16 \cdot 24$
19. 201^2
20. $75 \cdot 65$

APPLYING THE MATHEMATICS

In 21–25, tell whether the expression is a perfect square trinomial, difference of squares, or neither of these.

21. $u^2 - 2uj + j^2$
22. $9 - v^2$
23. $2sd + s^2 + d^2$
24. $xy - 16$
25. $-i^2 + p^2$
26. Solve $\frac{x-4}{7} = \frac{6}{x+4}$.
27. The numbers being multiplied in each part of Question 17 add to 60. Use the pattern found there to explain why, of all the pairs of numbers that add to 100, the largest product occurs when both numbers are 50.

In 28 and 29, expand and simplify the expression.

28. $(\sqrt{11} + \sqrt{13})(\sqrt{11} - \sqrt{13})$
29. $(3x + y)^2 + (3x - y)^2$

REVIEW

30. a. Expand $(x - 12)(x + 10)$.
 b. Solve $(x - 12)(x + 10) = 85$. (Lessons 11-6, 9-5)
31. After 7 years of putting money into a retirement account at a scale factor x , Lenny has saved $800x^6 + 1,000x^5 + 1,500x^4 + 1,200x^3 + 1,400x^2 + 1,800x + 2,000$ dollars. (Lesson 11-1)
- a. How much did Lenny put in during the most recent year?
 b. How much did Lenny put in during the first year?
 c. Give an example of a reasonable value for x , and evaluate the polynomial for that value of x .
32. Richard wants to construct a rectangular prism with height and width of x inches and length of 5 inches. He wants his prism to have the same volume as surface area. Construct a system with equations for the volume and surface area. Then solve for x . (Lesson 10-10)

In 33–35, describe a situation that might yield the given polynomial. (Lesson 8-2)

33. e^3

34. $6x^2$

35. $\pi r^2 - \pi s^2$

36. In 1965, Gordon Moore stated that computing speed in computers doubles every 24 months (Moore's Law). Computing speed is measured by transistors per circuit. (Lesson 7-2)
- a. In 1971, engineers could fit 4,004 transistors per circuit. Use Moore's Law to write an expression for the number of transistors per circuit that were possible in 1979.
- b. Many experts believe that Moore's Law will hold until 2020. Estimate the number of transistors per circuit possible in 2020, given that processors developed in 2000 had about 100 million transistors per circuit.

EXPLORATION

37. A CAS will be helpful in this question. After collecting terms, the expansion of $(a + b)^2$ has 3 unlike terms. Expand $(a + b + c)^2$. You should find that the expansion of $(a + b + c)^2$ has 6 unlike terms. How many unlike terms does the expansion of $(a + b + c + d)^2$ have? Try to generalize the result.

QY ANSWER

 $x^2 + 16x + 64$