Lesson

8-8 Solving Equations with Radicals

Vocabulary

extraneous solution

BIG IDEA To solve an equation of the form $x^{\frac{m}{n}} = k$, where m and n are integers, take the nth power of both sides. But be careful that you do not change the number of solutions to the equation in the process.

Remember that to solve an equation with a single rational power, such as $x^{\frac{4}{5}} = 10$, you can raise both sides to the power of the reciprocal of that exponent.

$$\left(x^{\frac{4}{5}}\right)^{\frac{5}{4}} = 10^{\frac{5}{4}}$$

So $x = 10^{\frac{5}{4}} \approx 17.78$. This checks because $17.78^{\frac{4}{5}} \approx 10$.

Mental Math

Find an equation for the inverse of the function and tell whether the inverse is a function.

a. $y = 7x$

b. $3x + y = 4.5$

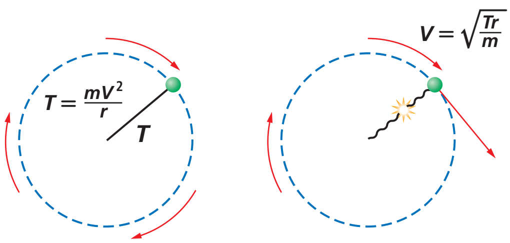
c. $y = 13$

Solving an Equation with a Single Radical

Similarly, because the radical $\sqrt[n]{\ }$ involves an nth root, you can solve an equation containing only this single radical by raising both sides to the nth power.

Example 1

Imagine spinning a ball on a string around in a circle. If the string breaks, the ball will follow a straight-line path in the direction it was traveling at the time of the break as shown below.



The ball will travel at a velocity V given by the formula $V = \sqrt{\frac{Tr}{m}}$, where T is the tension on the string (in newtons), m is the mass of the ball (in kilograms), and r is the length of the string (in meters). What tension is needed to allow a 5-kilogram ball on a 2-meter string to achieve a velocity of 10 $\frac{\text{meters}}{\text{second}}$?

Solution Here, $V = 10\ \frac{\text{m}}{\text{s}}$, $m = 5$ kg, and $r = 2$ meters. Substitute the given values into the formula and use a CAS to solve for T.

$$\sqrt{\frac{2 \cdot t}{5}} = 10 \qquad \frac{\sqrt{10 \cdot t}}{5} = 10$$

Enter the equation.

$$\left(\frac{\sqrt{10 \cdot t}}{5} = 10\right)^2 \qquad \frac{2 \cdot t}{5} = 100$$

Square both sides.

$$\left(\frac{2 \cdot t}{5} = 100\right) \cdot 5 \qquad 2 \cdot t = 500$$

Multiply each side by 5.

$$\frac{2 \cdot t = 500}{2} \qquad t = 250$$

Divide each side by 2.

The string tension needs to be 250 newtons.

Check Use the `solve` command. It checks.

$$\text{solve}\left(10 = \sqrt{\frac{t \cdot 2}{5}}, t\right) \qquad t = 250$$

STOP QY

Extraneous Solutions

There is a major difficulty that may occur when taking an nth power to solve equations with radicals. The new equation may have more solutions than the original equation does. So you must be careful to check each solution in the original equation. If a solution to a later equation does not check in the original equation, it is called an **extraneous solution**, and it is not a solution to the original equation.

QY

A 5-kilogram ball traveling 5 meters per second was attached to a string with a tension of 250 newtons when the string broke. How long was the string?

GUIDED

Example 2

Greg and Terrance both solve the equation $5 - \sqrt[6]{x} = 8$,and each one concludes that there are no real solutions. Complete Solution 1 to see Terrance's approach and Solution 2 to see Greg's.

Solution 1 Solve $5 - \sqrt[6]{x} = 8$.

$\underline{\ ?\ } = 3$ Add $\underline{\ ?\ }$ to both sides.

$(\underline{\ ?\ })^6 = (3)^6$ Raise both sides to 6th power.

$x = \underline{\ ?\ }$ Definition of $\underline{\ ?\ }$ power, arithmetic

Check Substitute $x = \underline{\ ?\ }$ into the original equation.

Does $5 - \sqrt[6]{\underline{\ ?\ }} = 8$?

$5 - \underline{\ ?\ } = 8$? No.

So, $\underline{\ ?\ }$ is not a solution. It is extraneous.

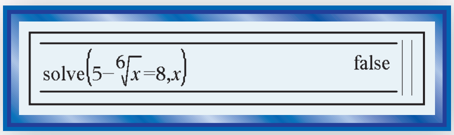
The sentence $5 - \sqrt[6]{x} = 8$ has $\underline{\ ?\ }$ real solutions.

Solution 2 Solve $5 - \sqrt[6]{x} = 8$.

$-\sqrt[6]{x} = 3$ Add –5 to both sides.

The left side of the equation is always a negative number because $\underline{\ ?\ }$. Therefore, the left side cannot equal positive 3. Thus, there are $\underline{\ ?\ }$ real solutions.

In real-number mode, a CAS solution to the equation in Example 2 is shown below. It means that there are no real solutions to this equation.



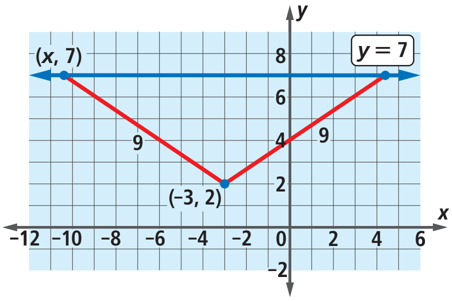
Equations from the Distance Formula

The Pythagorean Distance Formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ can lead to equations involving square roots. Although the equations may look quite complicated, they can be solved with the same approaches as simpler equations.

Example 3

Find coordinates for the two points on the line with equation $y = 7$ that are 9 units away from the point (-3, 2).

Solution Draw a picture like the one at the right. Let $(x, 7)$ be one of the points you want.



Because the distance from $(x, 7)$ to (-3, 2) is 9,

$$\sqrt{(x - (-3))^2 + (7 - 2)^2} = 9.$$

Simplify. $\sqrt{(x + 3)^2 + 25} = 9$

Now square both sides. $(x + 3)^2 + 25 = 81$

Subtract 25 from both sides. $(x + 3)^2 = 56$

Take the square root of both sides. $|x + 3| = \sqrt{56}$

So, $x + 3 = \sqrt{56}$ or $x + 3 = -\sqrt{56}$

$x = \sqrt{56} - 3$ or $x = -\sqrt{56} - 3$

$x \approx 4.5$ or $x \approx -10.5.$

The two points are exactly $(\sqrt{56} - 3, 7)$ and $(-\sqrt{56} - 3, 7)$, or approximately (4.5, 7) and (-10.5, 7).

Questions

COVERING THE IDEAS

1. Refer to Example 1. A 2-kilogram ball traveling at $8\frac{\text{m}}{\text{s}}$ is attached to a string with a tension of 100 newtons. How long is the string?

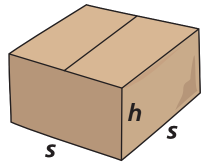
In 2–5, find all real solutions.

2. $\sqrt[3]{d} = 8$
3. $12 = 2\sqrt[4]{m}$
4. $25 + \sqrt[5]{g} = 10$
5. $3 - \frac{1}{2}\sqrt[6]{w} = -5$
6. What is an extraneous solution to an equation?

In 7 and 8, find all real solutions.

7. $\sqrt{8x - 3} = 4$
8. $12 - \sqrt[8]{3x - 1} = 15$
9. Find the two points on the line with equation $y = -2$ that are 4 units away from the point (1, -1).
10. Find the two points on the line with equation $y = x$ that are 3 units away from the point (5, 7).

APPLYING THE MATHEMATICS



11. Patty Packer is designing rectangular boxes like the one at the right, with square bases and a surface area of 18.2 square feet. The formula $s = -h + \sqrt{h^2 + 9.1}$ gives the base-side length s in terms of height h.

a. Find s when $h = 2$ feet.

b. Find h when $s = 2$ feet.

c. Derive the formula.

12. A sphere has a radius of r centimeters. A new sphere is created with a diameter 3 centimeters less than the original. If the new sphere has a volume of 1500 cm^3, what is the radius of the original sphere?

13. Recall the U.S. median home values from page 514. Below is a similar table with home-value data for the state of Texas from 1950 to 2000.

Year	1950	1960	1970	1980	1990	2000
Median Home Value (unadjusted dollars)	5805	8800	12,000	39,100	59,600	82,500

a. Calculate decade-to-decade percentage growth and the size-change factor for each of the five decades and record your results in a table as shown below. The first decade is done for you.

Years	1950–1960	1960–1970	1970–1980	1980–1990	1990–2000
% Increase	52	?	?	?	?
Size-change Factor	1.52	?	?	?	?

b. Compute the geometric mean of the size-change factors in Part a, and determine the average increase per decade, expressed as a whole percent, in Texas home values from 1950 to 2000.

c. The decade-to-decade percentage growth in Texas home values from 1940 to 2000 is ≈ 91%. If this continues, estimate the median Texas home value in 2010.

Ken Warby holds the world water speed record. His boat, the *Spirit of Australia*, traveled at 317.6 mph on October 8, 1978.

14. When traveling at a fast rate, a ship's speed s (in knots) varies directly as the seventh root of the power p (in horsepower) generated by the engine. Suppose the equation $s = 6.5\sqrt[7]{p}$ describes the situation for a particular ship. If the ship is traveling at a speed of 15 knots, about how much horsepower is the engine generating?

In 15 and 16, find all real solutions.

15. $5\sqrt[3]{x} - 8 = 13\sqrt[3]{x}$

16. $\sqrt{y^2 - 9} = 2\sqrt{y - 3}$

REVIEW

In 17 and 18, simplify each expression. Assume variables are nonnegative. (Lessons 8-7, 8-5)

17. $\sqrt[5]{-32y^{15}}$

18. $\sqrt{45a^5}\sqrt{5b^8}$

19. Give a counterexample to the statement $(x^4)^{\frac{1}{4}} = x$.

True or False In 20 and 21, assume $t > 0$. Justify your answer. (Lesson 8-6)

20. $\frac{1}{\sqrt{t}} = \frac{\sqrt{t}}{t}$

21. $\frac{\sqrt{3t}}{\sqrt{t}} = \frac{3\sqrt{t}}{\sqrt{3t}}$

22. **Multiple Choice** If a and k are positive, which of the following values for x is a solution to $a(x + h)^n = k$? (Lesson 8-6)

A $\left(\frac{a+h}{k}\right)^{-\frac{1}{n}}$

B $\sqrt[n]{\frac{k}{a}} - h$

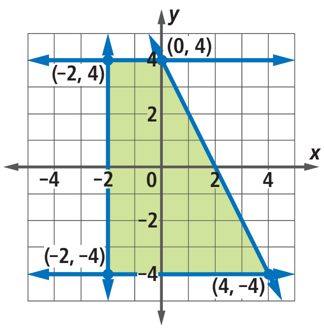
C $\sqrt[n]{\frac{k-a}{h}}$

D $\sqrt[n]{\frac{k}{a}} + h$

23. Let $u(x) = \sqrt[3]{x}$ and $v(x) = x^6$. (Lessons 8-4, 8-1)
    a. Find an equation for $u \circ v$. What is the domain of $u \circ v$?
    b. Find an equation for $v \circ u$. What is the domain of $v \circ u$?

24. Explain why the inverse of the function $f: x \rightarrow x(x + 2)$ is not a function. (Lessons 8-3, 8-2)

25. Write a system of inequalities whose solution is the shaded region graphed at the right. (Lesson 5-8)



26. In 2000, one estimate of known oil reserves worldwide was 1.017×10^{12} barrels (1020 gigabarrels) while annual consumption was estimated to be 2.80×10^{10} barrels. If oil consumption remains constant, how many years after 2000 will known oil reserves last? (Lesson 7-2)

EXPLORATION

27. If you flip a fair coin n times, the expected difference between the number of heads and tails is the integer closest to $\sqrt{\frac{2n}{\pi}}$.
    a. How many times would you have to flip a coin to get an expected difference of 8?
    b. Test this formula by using a spreadsheet to simulate flipping a coin the number of times you calculated in Part a. Do multiple simulations and create a bar chart. Does the formula appear to work for your value of n?

QY ANSWER

$\frac{1}{2}$ m