Lesson

8-7 Powers and Roots of Negative Numbers

Vocabulary

$\sqrt[n]{x}$ when $x < 0$

BIG IDEA Great care must be taken when dealing with *n*th roots of negative numbers. When the *n*th roots are not real, then the Root of a Product Theorem may not be true.

Mental Math

Miguel is training to run a 3-mile race. In how much time must he run each mile on average if he wants to finish the race in

a. 30 minutes?

b. 28 minutes?

c. 26 minutes?

You have already calculated some powers and some square roots of negative numbers. First we review the powers.

Integer Powers of Negative Numbers

Activity

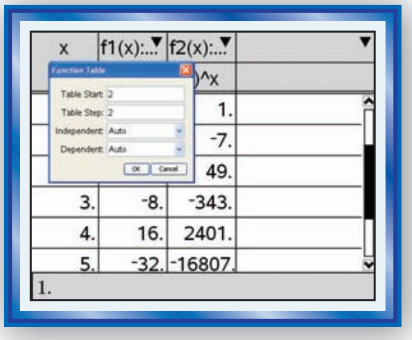
Work in pairs. Assign one pair of functions to each partner.

a. $y = (-2)^x$
$y = (-7)^x$

b. $y = (-6)^x$
$y = (-5)^x$

Step 1 Enter your functions into a calculator and generate a table of values starting at $x = 2$ with an increment of 2.

Step 2 Scroll through the table of values. Compare the results for all four functions with your partner. Describe any patterns you see.



Step 3 Generate a table of values starting at $x = 1$ with an increment of 2. Scroll through the table of values. Compare the results for all four functions with your partner. Describe any patterns you see.

Step 4 Complete the following generalization, choosing the correct words: __?__ (Even/Odd) integer exponents with negative bases produce __?__ (positive/negative) powers, while __?__ (even/odd) integer exponents with negative bases produce __?__ (positive/negative) powers.

The Activity shows that positive integer powers of negative numbers alternate between positive and negative numbers. Even exponents produce positive numbers, while odd exponents produce negative numbers. The same is true if zero and negative powers are considered, because $(-x)^{-n}$ is the reciprocal of $(-x)^n$.

Integer powers of negative numbers satisfy the order of operations. For example, $-6^4 = -1296$ because the power is calculated before taking the opposite. However, $(-6)^4 = 1296$, so $-6^4 \neq (-6)^4$. All the properties of *integer* powers of positive bases that you studied in Chapter 7 also apply to integer powers of negative bases.

GUIDED

Example 1

Write without exponents: $(-3)^5(-3)^{-2}$.

Solution Use the Product of Powers Postulate.

$(-3)^5(-3)^{-2} = (-3)^{\underline{?} + \underline{?}} = (-3)^{\underline{?}} = \underline{?}$

Are There Noninteger Powers of Negative Numbers?

Many times in this course, you have seen that powers and roots of negative numbers do not have the same properties that powers and roots of positive numbers do. Here are some of the properties that are different.

(1) When x is positive, $\sqrt{x}$ is a real number, but $\sqrt{-x}$ is a pure imaginary number.

(2) If both x and y are negative, $\sqrt{x} \cdot \sqrt{y} \neq \sqrt{xy}$. The left side is the product of two imaginary numbers and is negative; the right side is positive. For example, $\sqrt{-3} \cdot \sqrt{-2} \neq \sqrt{(-3)(-2)}$ because $i\sqrt{3} \cdot i\sqrt{2} = -\sqrt{6}$ and $\sqrt{(-3)(-2)} = \sqrt{6}$.

(3) If x is negative, then $x^n \neq \sqrt{x^{2n}}$ for positive integers n. Again, the left side is negative and the right side is positive. For example, $(-2)^3 \neq \sqrt{(-2)^6}$ because $(-2)^3 = -8$ and $\sqrt{(-2)^6} = \sqrt{64} = 8$.

These examples indicate that powers and roots of negative numbers have to be dealt with very carefully. For this reason, we do not define x^m when x is negative and m is not an integer. *Noninteger powers of negative numbers are not defined in this book*. That is, an expression such as $(-3)^{\frac{1}{2}}$ is not defined. However, we allow square roots of negative numbers to be represented by a radical.

STOP **QY**

QY

Explain why $\sqrt{-5} \cdot \sqrt{-11} \neq \sqrt{-55}$.

The Expression $\sqrt[n]{x}$ When x Is Negative and n Is Odd

When x is positive, the radical symbol $\sqrt[n]{x}$ stands for its unique positive nth root. It would be nice to use the same symbol for an nth root of a negative number. This can be done for odd roots of negative numbers. If a number is negative, then it has exactly one real odd root. For instance, –27 has one real cube root, namely –3. Consequently, it is customary to use the symbol $\sqrt[n]{x}$ when x is negative, provided n is odd.

Definition of $\sqrt[n]{x}$ when $x < 0$

When x is negative and n is an odd integer > 2, $\sqrt[n]{x}$ stands for the real nth root of x.

For instance, because $(-5)^3 = -125$, $\sqrt[3]{-125} = -5$.
Because $-100{,}000 = (-10)^5$, you can write $\sqrt[5]{-100{,}000} = -10$.

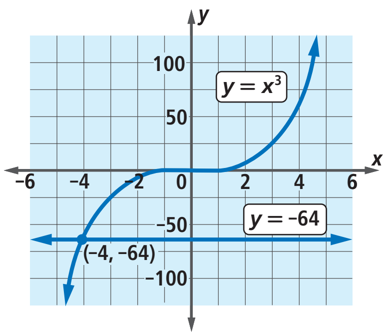
To evaluate nth roots of negative numbers without a calculator, you can use numerical or graphical methods.

Example 2

Evaluate $\sqrt[3]{-64}$.

Solution 1 $\sqrt[3]{-64}$ represents the real 3rd root of –64, so you can solve $x^3 = -64$. Because $(-4)^3 = -64$, $\sqrt[3]{-64} = -4$.

Solution 2 Graph $y = x^3$ and $y = -64$. The x-coordinate of the point of intersection is the real 3rd root of –64, as shown at the right. So, $\sqrt[3]{-64} = -4$.



Notice that the graph of the function f with equation $y = x^3$ or $f(x) = x^3$ verifies that every real number has exactly one real 3rd root, because any horizontal line intersects the graph only once. Therefore, –4 is the only real 3rd root of –64.

The Expression $\sqrt[n]{x}$ When x Is Negative and n Is Even

Square roots of negative numbers are not real numbers, and they do not satisfy all the properties of square roots of positive numbers. However, the radical form $\sqrt{x}$ is used when $x < 0$. When x is negative, $\sqrt{x} = i\sqrt{-x}$. For example, $\sqrt{-7} = i\sqrt{7}$. Other even roots of negative numbers (4th roots, 6th roots, 8th roots, and so on) are also not real, but they are not written using radicals. So, the nth-root expression $\sqrt[n]{x}$ is not defined when x is negative and n is an even integer greater than 2.

Here is a summary of our use of the $\sqrt[n]{}$ symbol when $n > 2$:

(1) When $x \geq 0$, $\sqrt[n]{x}$ is defined for any integer $n > 2$. It equals the positive real nth root of x.

(2) When $x < 0$, $\sqrt[n]{x}$ is defined only for odd integers $n \geq 3$. It equals the negative real nth root of x.

This summary may seem unnecessarily detailed, but it allows you to handle expressions with radical signs in much the same way that square roots are handled, as long as the expressions stand for real numbers.

nth Root of a Product Theorem

When $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are defined and are real numbers, then $\sqrt[n]{xy}$ is also defined and $\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}$.

Example 3

Simplify $\sqrt[5]{-640}$. Leave your answer in radical form.

Solution Look for a perfect fifth power that is a factor of –640.

$-640 = -32 \cdot 20 = (-2)^5 \cdot 20$

So, $\sqrt[5]{-640} = \sqrt[5]{-32} \cdot \sqrt[5]{20}$

$= -2 \cdot \sqrt[5]{20}$.

Questions

COVERING THE IDEAS

1. Let $f(x) = (-9)^x$.
   a. Give three values of x that produce a positive value of $f(x)$.
   b. Give three values of x that produce a negative value of $f(x)$.

2. Calculate $(-8)^n$ for all integer values of n from -3 to 3.

3. Tell whether the number is positive or negative.

   a. $(-3)^4$ b. -3^4 c. $(-4)^{-3}$ d. $(-4)^3$

4. **True or False**

   a. $(-x)^{10} = -x^{10}$ b. $(-x)^9 = -x^9$

In 5 and 6, write as a single power.

5. $(-2)^6(-2)^{-3}$

6. $((-4)^5)^6$

7. Calculate $\sqrt{-4} \cdot \sqrt{-9}$.

In 8–10, evaluate.

8. $\sqrt[3]{-125x^9}$

9. $\sqrt[11]{-1}$

10. $\sqrt[9]{-512 \cdot 10^{27}}$

In 11–13, simplify.

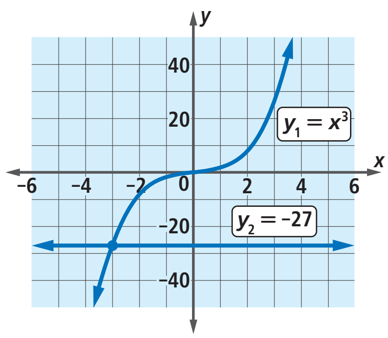
11. $\sqrt[3]{-\frac{64y^{27}}{27}}$

12. $\sqrt[9]{-10^{63}}$

13. $\sqrt[7]{1280q^{23}}$

14. What are the domain and the range of the real function with equation $y = \sqrt[6]{x}$?

15. The graphs of $y_1 = x^3$ and $y_2 = -27$ are shown below. What is the significance of the point of intersection of y_1 and y_2?



APPLYING THE MATHEMATICS

16. Simplify.

   a. $\sqrt[5]{-32} + \sqrt[4]{16}$ b. $\sqrt[5]{-32p^{10}} + \sqrt[4]{16p^{16}}$

17. Explain why the graphs of $y = \sqrt[3]{x}$ and $y = \sqrt[18]{x^6}$ are not the same.

18. Let $f : x \rightarrow \sqrt[5]{x}$ and $g : x \rightarrow \sqrt[15]{x}$. Find $f \circ g(x)$.

19. a. Show that $2 + 2i$ is a 4th root of -64.

   b. Show that $-2 - 2i$ is a 4th root of -64.

   c. Show that $2 - 2i$ is a 4th root of -64.

   d. Find the one other 4th root of -64 and verify your finding.

   e. How are Parts a–d related to the fact that $\sqrt[4]{-64}$ is not defined?

REVIEW

In 20 and 21, rationalize the denominator and simplify. (Lesson 8-6)

20. $\frac{5}{\sqrt{7}}$

21. $\frac{2 - \sqrt{3}}{2 + \sqrt{3}}$

22. a. Find the geometric mean of 2, 4, 8, 16, and 32.
    b. Generalize Part a. (Lesson 8-4)

23. Evaluate $\sqrt[3]{\sqrt{4096}}$. (Lesson 8-4)

24. a. Simplify without using a calculator: $(\sqrt{7} - \sqrt{13})(\sqrt{7} + \sqrt{13})$.
    b. Check by approximating $\sqrt{7}$ and $\sqrt{13}$ by decimals with a calculator and multiplying the decimals. (Lessons 8-5, 6-2)

25. Solve $r^{-\frac{2}{3}} = 64^{-1}$ for r. (Lesson 7-8)

26. A rectangle has vertices at (2, 0), (6, 4), (4, 6), and (0, 2). What is its area? (Previous Course)

In 27 and 28, use the fact that the population in the United States was about 2.96×10^8 in 2005. (Lesson 7-2)

27. With a land area of about 3.5×10^6 mi^2, what was the average number of people per square mile?

28. In 2005, people in the U.S. consumed about 2.78×10^{10} pounds of beef. About how much beef was consumed per person in the U.S. in 2005?

EXPLORATION

29. Use your results from Part a to answer the other parts.
    a. Sketch the graphs of $y = \sqrt[3]{x^3}$, $y = \sqrt[4]{x^4}$, $y = \sqrt[5]{x^5}$, and $y = \sqrt[6]{x^6}$.
    b. For what values of n does $\sqrt[n]{x^n} = x$ for every real number x?
    c. For what values of n does $\sqrt[n]{x^n} = |x|$ for every real number x?
    d. Does $y = \sqrt{x^2}$ follow the pattern of other nth root of nth power functions?

QY ANSWER

$\sqrt{-5} \cdot \sqrt{-11}$ is a negative number, but $\sqrt{-55}$ is an imaginary number.