Lesson 8-6 Quotients with Radicals

Vocabulary

rationalizing the denominator

conjugate

▶ **BIG IDEA** A fraction with a denominator of the form $a + b\sqrt{c}$ can be rewritten without any square root in the denominator.

In the book *The Phantom Tollbooth* (Juster, 1961), Milo and his traveling companions Humbug and Tock encounter the sign below, which gives the distance to the land of Digitopolis.

DIGITOPOLIS	
5	Miles
1,600	Rods
8,800	Yards
26,400	Feet
316,800	Inches
633,600	Half Inches
AND THEN SOME	

The characters argue about which distance to travel. Humbug wants to use miles because he believes the distance is shorter, while Milo wants to travel by half inches because he thinks it will be quicker.

The joke is that all the distances are the same. For example, 1 rod (a measure used in surveying) is equal to 16.5 ft, so $\frac{26{,}400 \text{ ft}}{16.5 \frac{\text{ft}}{\text{rod}}} = 1600$ rods. The point is that there are many ways to express one quantity or one number. You saw this when simplifying radicals in the previous lesson. Now we apply this idea to find different ways of writing quotients with radicals.

Mental Math

At a certain time of day, a tree casts an 18 ft shadow.

a. A nearby 8 ft light pole casts a 12 ft long shadow. How tall is the tree?

b. How long is the shadow cast by the nearby 3 ft mailbox?

c. When Trey gets the mail, his shadow is twice as long as the mailbox's shadow. How tall is he?

Rationalizing When the Denominator Is a Radical

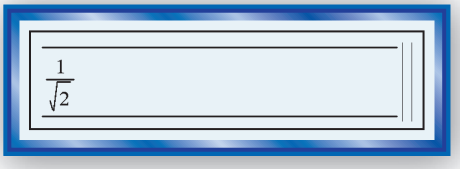
Think about how you might approximate the value of $\frac{1}{\sqrt{2}}$ without a calculator. Using long division is no help because dividing 1 by $\sqrt{2}$ requires you to calculate $1.414213\ldots\overline{)1.00}$, which cannot be done by hand. Instead, a process called **rationalizing the denominator** is used to write an equivalent form of the number without a radical in the denominator. *Rationalizing* means rewriting the fraction so that its denominator is a rational number.

Activity

MATERIALS CAS

Step 1 Enter the following expressions into a CAS and record the results. Do not use decimal approximations.

a. $\frac{1}{\sqrt{2}}$ **b.** $\frac{3}{\sqrt{5}}$ **c.** $\frac{13}{\sqrt{7}}$



Step 2 Approximate each original expression in Step 1 and the resulting CAS expression to 3 decimal places. Do the pairs of numbers appear to be equal?

Step 3 Rewrite $\frac{1}{\sqrt{a}}$ $(a > 0)$ without a radical in the denominator.

The results of the Activity suggest a method for rationalizing denominators of fractions whose denominators are square roots. In general, $\frac{a}{\sqrt{x}} = \frac{a}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{a\sqrt{x}}{x}$ for $x > 0$. This works because $\frac{\sqrt{x}}{\sqrt{x}} = 1$ and $\sqrt{x} \cdot \sqrt{x} = x$ for all real numbers x.

STOP QY

Because of technology, rationalizing denominators to obtain close approximations of quotients is no longer necessary. However, rationalizing denominators is still a useful process because not all technologies put results with radicals in the same form. You can expect to see different but equivalent forms of rationalized expressions depending upon the technology you use.

You can also rationalize denominators involving variable expressions.

QY

Rewrite $\frac{26}{\sqrt{13}}$ by rationalizing the denominator.

GUIDED

Example 1

Rationalize the denominator of $\frac{3}{\sqrt{12x}}$, where $x > 0$.

Solution 1 Simplify the denominator first, then rationalize.

$$\frac{3}{\sqrt{12x}} = \frac{3}{\sqrt{3 \cdot 4 \cdot x}} = \frac{3}{\sqrt{3} \cdot \sqrt{?} \cdot \sqrt{?}} = \frac{3}{? \cdot \sqrt{?}} \cdot \frac{?}{?} = \frac{?}{?}$$

Solution 2 Rationalize first, then simplify.

$$\frac{3}{\sqrt{12x}} \cdot \frac{?}{?} = \frac{?}{12x} = \frac{?}{?}$$

Rationalizing When the Denominator Is a Sum Containing a Radical

Now consider a fraction $\frac{n}{a + \sqrt{b}}$, in which a radical in the denominator is added to another term. In this form you cannot easily separate the rational and irrational parts. However, to rationalize the denominator, you can use a technique similar to the one you used in Lesson 6-9 to divide complex numbers. Recall that to write $\frac{1}{5 + 2i}$ in $a + bi$ form, you multiply both numerator and denominator by the complex conjugate of the denominator, $5 - 2i$. To rationalize a fraction with a denominator of the form $a + \sqrt{b}$, multiply both numerator and denominator by the **conjugate** $a - \sqrt{b}$. The product has a denominator with no radical terms, because $(x + y)(x - y) = x^2 - y^2$.

▸ READING MATH

The word *conjugate* comes from the Latin prefix *co-* meaning "together with," and the Latin verb *jugare*, meaning "to join" or "to connect." In algebra, two complex numbers or radical expressions that are conjugates are joined together as a pair.

Example 2

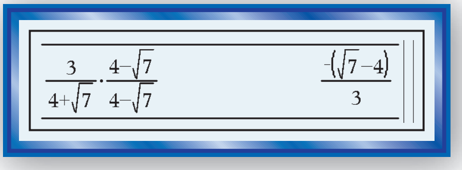
Write $\frac{3}{4 + \sqrt{7}}$ in $a + b\sqrt{c}$ form.

Solution 1 The conjugate of $4 + \sqrt{7}$ is $4 - \sqrt{7}$.

$$\frac{3}{4 + \sqrt{7}} \cdot \frac{4 - \sqrt{7}}{4 - \sqrt{7}} = \frac{3(4 - \sqrt{7})}{4^2 - \sqrt{7}^2} = \frac{3(4 - \sqrt{7})}{16 - 7} = \frac{3(4 - \sqrt{7})}{9} = \frac{4 - \sqrt{7}}{3}$$

$$= \frac{4}{3} - \frac{\sqrt{7}}{3}$$

Solution 2 Use a calculator to multiply the numerator and denominator by the conjugate. The result can be rewritten in $a + b\sqrt{c}$ form as $\frac{4}{3} - \frac{\sqrt{7}}{3}$.



Check Estimate the original and final expressions with decimals.

$$\frac{3}{4 + \sqrt{7}} \approx 0.45142$$

$$\frac{4}{3} - \frac{\sqrt{7}}{3} \approx 0.45142$$

It checks.

Questions

COVERING THE IDEAS

1. Verify that the indicated numbers on the DIGITOPOLIS sign are equivalent.
   a. the number of inches and the number of yards
   b. the number of yards and the number of miles
2. Why is it impossible to do long division with $\sqrt{73}$ as a divisor?

3. Estimate the value of $\frac{\sqrt{2}}{2}$ to the nearest tenth.

4. What does the term *rationalize the denominator* mean?

5. If $a > 0$, rewrite $\frac{b}{\sqrt{a}}$ without a radical in the denominator.

6. Are $\frac{2}{\sqrt{17}}$ and $\frac{2\sqrt{17}}{17}$ equal? Justify your answer.

In 7 and 8, rationalize the denominator.

7. $\frac{11}{\sqrt{3}}$

8. $\frac{4}{\sqrt{34}}$

In 9 and 10, a fraction is given.

a. Tell what you would multiply the fraction by to rationalize the denominator.

b. Rationalize the denominator and write the result in $a + b\sqrt{c}$ form.

9. $\frac{7}{3 - \sqrt{5}}$

10. $\frac{16}{\sqrt{12t} + 8}$

APPLYING THE MATHEMATICS

In 11–15, rationalize the denominator of each expression. Assume all variables are positive.

11. $\frac{3}{2\sqrt{3}}$

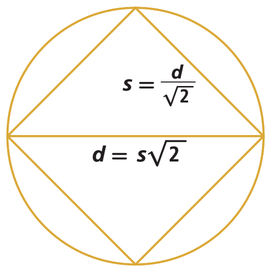
12. $\frac{47}{\sqrt{47}}$

13. $\frac{3x}{\sqrt{9x^5}}$

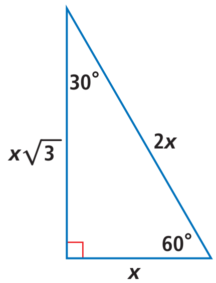
14. $\frac{5 - \sqrt{12}}{5 + \sqrt{12}}$

15. $\frac{4}{\sqrt{n} - 6}$

16. As pictured at the right, the largest square piece of wood that can be cut out of a circular log with diameter d has a side of length $\frac{d}{\sqrt{2}}$. If the radius of a log is 17 in., what is the side length of the largest square piece of wood that can be cut from it? Give your answer in both rationalized form and to the nearest hundredth of an inch.



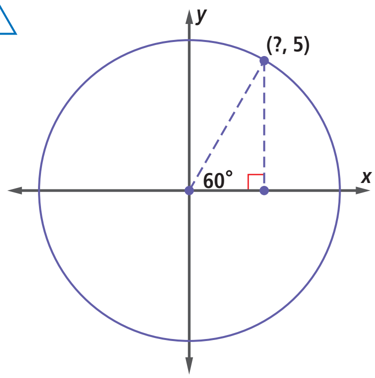
17. Recall from geometry the ratio of the sides of a 30-60-90 triangle as shown in the diagram at the right.



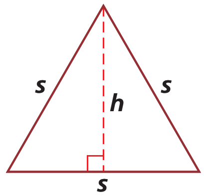
a. If the length of the longer leg of the triangle is 8, find the length of the hypotenuse in rationalized form.

b. If the length of the longer leg of the triangle is a, write rationalized expressions for the lengths of the other two sides of the triangle in terms of a.

c. What is the missing x-coordinate of the point on the circle at the far right? Rationalize the denominator of your answer.



18. In the equilateral triangle at the right, find the ratio of s to h. Write your answer with a rationalized denominator.



REVIEW

In 19 and 20, simplify the expression. Assume *a* and *b* are nonnegative real numbers. (Lesson 8-5)

19. $\sqrt[3]{54a^7}$

20. $\sqrt[4]{64a^3b^2}\sqrt{8a^6b}$

True or False In 21 and 22, assume all variables are positive. Justify your answer. (Lesson 8-4)

21. $\sqrt{xy^{\frac{1}{4}}} = \sqrt[8]{x^4y}$

22. $\sqrt[3]{m^{\frac{1}{4}}n} = \sqrt[36]{m^3n^{12}}$

23. Suppose $g : x \rightarrow x^{\frac{1}{3}}$, and $x > 0$. (Lessons 8-3, 8-2, 8-1)
   a. Find an equation for g^{-1}.
   b. If $h : x \rightarrow 2x$, find $g \circ h(x)$.

24. Explain why the inverse of $f : x \rightarrow 7x^{38}$ is not a function. (Lessons 8-2, 7-1)

25. Let f be a function defined by the table below. Is the inverse of f a function? Why or why not? (Lesson 8-2)

x	1	3	4	8	11	12
$f(x)$	3	5	13	5	–2	4

In 26 and 27, assume *p* and *q* are positive numbers. Write each expression without negative exponents. (Lesson 7-8)

26. $p^{-\frac{1}{5}}q^{\frac{3}{8}}$

27. $\dfrac{p^{-\frac{2}{3}}}{p}$

In 28 and 29, write an expression to describe the situation. (Lesson 1-1)

28. Kim collects teacups. She now has 14 teacups and buys one new teacup per month. How many teacups will she have after m months?

29. You buy G granola bars at d dollars per bar. How much did you spend?

EXPLORATION

30. A friend tries to rationalize $\frac{1}{\sqrt[3]{x}}$ by performing the following multiplication:
$$\frac{1}{\sqrt[3]{x}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}}.$$
   a. Explain why this method will not work.
   b. Devise a method to rationalize the denominator of $\frac{1}{\sqrt[3]{x}}$.
   c. Rewrite $\frac{1}{\sqrt[3]{x}}$ as a rational power of x.

QY ANSWER

$\frac{26}{\sqrt{13}} = \frac{26}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = \frac{26\sqrt{13}}{13}$
$= 2\sqrt{13}$