Lesson 8-5 Products with Radicals

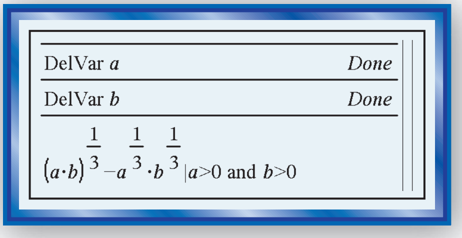
BIG IDEA The product of the *n*th roots of nonnegative numbers is the *n*th root of the product of the numbers.

Activity

MATERIALS CAS

Clear variables *a* and *b* and set the CAS to real-number mode.

Step 1 Enter the following expressions, with $a > 0$ and $b > 0$, into the CAS and record the results.



a. $(a \cdot b)^{\frac{1}{3}} - a^{\frac{1}{3}} \cdot b^{\frac{1}{3}}$

b. $(a \cdot b)^{\frac{1}{12}} - a^{\frac{1}{12}} \cdot b^{\frac{1}{12}}$

c. $a^{\frac{1}{4}} \cdot b^{\frac{1}{4}} - (a \cdot b)^{\frac{1}{4}}$

d. $a^{\frac{1}{7}} \cdot b^{\frac{1}{7}} - (a \cdot b)^{\frac{1}{7}}$

Step 2 Based on the results from Step 1, make a conjecture:
$(a \cdot b)^{\frac{1}{n}} - a^{\frac{1}{n}} \cdot b^{\frac{1}{n}} = \underline{\ ?\ }$.

Step 3 Based on your conjecture in Step 2, make another conjecture:
$(a \cdot b)^{\frac{1}{n}} = \underline{\ ?\ }$.

The results of the Activity suggest a property of *n*th roots. This property can also be derived another way. Recall that for all nonnegative numbers *a* and *b*,

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

Rewriting the above equation with rational powers instead of radicals you can see that this property of radicals is a special case of the Power of a Product Postulate, $(ab)^m = a^m \cdot b^m$.

$$(ab)^{\frac{1}{2}} = a^{\frac{1}{2}} \cdot b^{\frac{1}{2}}$$

If you now let $m = \frac{1}{n}$ in the Power of a Product Postulate, you obtain a theorem about the product of *n*th roots.

Mental Math

Simplify.

a. $\frac{w^{17}}{w^{12}}$

b. $\frac{x^3}{x^6} \cdot x^5$

c. $4y\left(\frac{3y}{y^4}\right)^2$

d. $\left(8wz^2 + \frac{z^3}{2w^2}\right)w^3$

Root of a Product Theorem

For any nonnegative real numbers x and y, and any integer $n \geq 2$,

$(xy)^{\frac{1}{n}} = x^{\frac{1}{n}} \cdot y^{\frac{1}{n}}$. power form

$\sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y}$. radical form

Multiplying Radicals

You can use the Root of a Product Theorem to multiply nth roots.

Example 1

Calculate $\sqrt[3]{50} \cdot \sqrt[3]{20}$ without a calculator.

Solution Rewrite the given expression using the Root of a Product Theorem.

$\sqrt[3]{50} \cdot \sqrt[3]{20} = \sqrt[3]{50 \cdot 20}$ Root of Product Theorem

$= \sqrt[3]{1000}$ Arithmetic

$= 10$ Definition of cube root

Check Use a calculator to find 3-place decimal approximations for $\sqrt[3]{50}$ and $\sqrt[3]{20}$.

$\sqrt[3]{50} \approx 3.684$ $\sqrt[3]{20} \approx 2.714$

Multiply the decimals.

$3.684 \cdot 2.714 = 9.998376$, close enough given the estimates.

GUIDED

Example 2

Assume that $x \geq 0$. Perform the multiplication: $\sqrt[4]{5x} \cdot \sqrt[4]{125x^3}$.

Solution 1 Rewrite using the Root of a Product Theorem.

$\sqrt[4]{5x} \cdot \sqrt[4]{125x^3} = \sqrt[4]{\underline{\ ?\ }}$

Now use the Root of a Product Theorem to rewrite again.

$= \sqrt[4]{\underline{\ ?\ }} \cdot \sqrt[4]{\underline{\ ?\ }}$

$= \underline{\ ?\ }$

Solution 2 Convert to rational exponents.

$$\sqrt[4]{5x} \cdot \sqrt[4]{125x^3} = (5x)^{\frac{1}{4}} \cdot (\underline{\ ?\ })^{\frac{1}{4}}$$
$$= (\underline{\ ?\ })^{\frac{1}{4}}$$
$$= 625^{\frac{1}{4}} \cdot (\underline{\ ?\ })^{\frac{1}{4}}$$
$$= 5x$$

Simplifying Radicals

In Example 2, you used the Root of a Product Theorem to rewrite an *n*th root as a product. For instance, $\sqrt[3]{240}$ can be rewritten several ways:

$$\sqrt[3]{240} = \sqrt[3]{2 \cdot 120} = \sqrt[3]{2} \cdot \sqrt[3]{120};$$
$$\sqrt[3]{240} = \sqrt[3]{8 \cdot 30} = \sqrt[3]{8} \cdot \sqrt[3]{30};$$
$$\sqrt[3]{240} = \sqrt[3]{16 \cdot 15} = \sqrt[3]{16} \cdot \sqrt[3]{15}.$$

Because 8 is a perfect cube, the second form shows that $\sqrt[3]{240} = 2\sqrt[3]{30}$. Some people call $2\sqrt[3]{30}$ the ***simplified form*** of $\sqrt[3]{240}$. In general, to simplify an *n*th root, rewrite the expression under the radical sign as a product of perfect *n*th powers and other factors. Then apply the Root of a Product Theorem.

Example 3

Suppose $r \geq 0$ and $s \geq 0$. Simplify the expression $\sqrt[3]{64r^6s^{15}}$.

Solution 1 Because it is a third root, identify perfect third powers in the expression under the radical.

$$64 = 4^3,\ r^6 = (r^2)^3, \text{ and } s^{15} = (s^5)^3$$

Rewrite and simplify.

$\sqrt[3]{64r^6s^{15}}$	$= \sqrt[3]{4^3(r^2)^3(s^5)^3}$	Power of a Power Property
	$= \sqrt[3]{4^3} \cdot \sqrt[3]{(r^2)^3} \cdot \sqrt[3]{(s^5)^3}$	Root of a Product Theorem
	$= 4r^2s^5$	Root of a Power Theorem

Solution 2 Rewrite using rational exponents.

$\sqrt[3]{64r^6s^{15}}$	$= (64r^6s^{15})^{\frac{1}{3}}$	Definition of $\sqrt[n]{x}$
	$= 64^{\frac{1}{3}} \cdot r^{\frac{6}{3}} \cdot s^{\frac{15}{3}}$	Power of a Product and Power of a Power Postulates
	$= 4r^2s^5$	Arithmetic

STOP **QY1**

QY1

Suppose $a \geq 0$ and $b \geq 0$. Simplify $\sqrt{16a^4b^{10}}$.

Sometimes when you try to simplify a radical, some irreducible portions remain, as in Example 4. Then the new expression may be more complicated than the given expression.

Example 4

Suppose $x \geq 0$. Rewrite $\sqrt[3]{120x^4}$ with a smaller power of x inside the radical.

Solution 1 Keep as a radical.

$$\sqrt[3]{120x^4} = \sqrt[3]{2^3 \cdot 15 \cdot x^3 \cdot x}$$
$$= \sqrt[3]{2^3 \cdot x^3} \cdot \sqrt[3]{15x}$$
$$= 2x\sqrt[3]{15x}$$

Solution 2 Convert to rational exponents.

$$\sqrt[3]{120x^4} = (120x^4)^{\frac{1}{3}}$$
$$= 120^{\frac{1}{3}} \cdot x^{\frac{4}{3}}$$
$$= 8^{\frac{1}{3}} \cdot 15^{\frac{1}{3}} \cdot x \cdot x^{\frac{1}{3}}$$
$$= 2x(15x)^{\frac{1}{3}}$$

STOP QY2

QY2

Suppose $w \geq 0$. Rewrite $\sqrt[5]{w^{17}}$ with a smaller power of w inside the radical.

Questions

COVERING THE IDEAS

1. State the Root of a Product Theorem.

2. **True or False** $\sqrt{50} \cdot \sqrt{3} = \sqrt{15} \cdot \sqrt{10}$

In 3 and 4, multiply and simplify.

3. $\sqrt[4]{1000} \cdot \sqrt[4]{100{,}000}$

4. $\sqrt[3]{9} \cdot \sqrt[3]{81}$

5. Write three different expressions equal to $\sqrt[3]{250}$.

In 6 and 7, find a and b. Assume $a > 0$ and $b > 0$.

6. $\sqrt{360} = \sqrt{a} \cdot \sqrt{10} = b\sqrt{10}$

7. $\sqrt[3]{297} = \sqrt[3]{a} \cdot \sqrt[3]{11} = b\sqrt[3]{11}$

In 8 and 9, simplify the radicals.

8. $\sqrt[3]{1250}$

9. $\sqrt{98} \cdot \sqrt{14}$

In 10–12, assume all variables are nonnegative. Simplify or rewrite with a smaller power of the variable inside the radical.

10. a. $\sqrt{144x^4}$ b. $\sqrt{144x^5}$

11. a. $\sqrt[3]{64y^{18}}$ b. $\sqrt[3]{\frac{y^{25}}{8}}$

12. $\sqrt[4]{1250y^3p^{17}}$

APPLYING THE MATHEMATICS

In 13 and 14, simplify the expression.

13. $\sqrt[3]{12} \cdot \sqrt[3]{18}$

14. $\sqrt[4]{144 \cdot 10^3} \cdot \sqrt[4]{9 \cdot 10^5}$

In 15 and 16, identify which expression is greater.

15. $\sqrt[3]{5} + \sqrt[3]{5}$ or $\sqrt[3]{10}$

16. $\sqrt[3]{600{,}000}$ or 100

17. Ali Baster simplifies $\sqrt[3]{4} \cdot \sqrt[6]{5}$ to $\sqrt[6]{20}$.
   a. Why is Ali's result incorrect?
   b. Solve $\sqrt[3]{4} \cdot \sqrt[6]{5} = \sqrt[6]{n}$ for n.

In 18 and 19, assume all variables are positive. Rewrite with a simpler expression inside the radical.

18. $\sqrt[3]{640x^{12}y^{11}}$

19. $\sqrt{4x^2 + 4y^2}$

REVIEW

20. Since 2000, the price of gasoline has been quite volatile. Here are the changes in average price for the years 2000–2006. Follow the steps to find the average annual price increase.

Year	Percent Increase From Preceding Year	Size-Change Factor
2000	28.5	?
2001	–3.6	?
2002	–6.5	?
2003	16.5	?
2004	18.2	?
2005	21.9	?
2006	12.9	?

   a. Fill in the table above by converting each percent into a size-change factor. **(Previous Course)**
   b. Compute the geometric mean of the factors you found in Part a to determine the average annual percentage increase. **(Lesson 8-4)**

21. Suppose $f(x) = \sqrt[4]{x}$. (Lesson 8-4)
 a. If x increases from 1 to 2, by how much does $f(x)$ increase?
 b. If x increases from 11 to 12, by how much does $f(x)$ increase?

22. **True or False** The inverse of the power function f with $f(x) = x^a$ is $f^{-1}(x) = x^{-a}$. Explain your reasoning. (Lesson 8-3)

23. a. Find an equation for the inverse of the linear function L defined by $L(x) = mx + b$.
 b. How are the slopes of the function and its inverse related?
 c. When is the inverse of L not a function? (Lessons 8-2, 3-1)

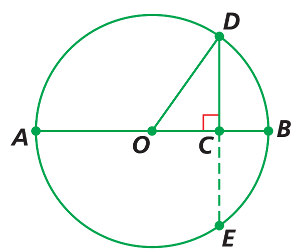
24. Solve for real values of y and check: $\frac{y^{-3}}{y} = \frac{1}{81}$. (Lesson 7-3)

25. Write $\frac{2 + 3i}{7 - 6i}$ in $a + bi$ form. (Lesson 6-9)

26. a. Multiply $\left(2 - \sqrt{5}\right) \cdot \left(20 + 10\sqrt{5}\right)$. (Lesson 6-1)
 b. Your answer to Part a should be an integer. Tell how you could know that in advance.

EXPLORATION

27. The diagram at the right can be used to compare the arithmetic mean and geometric mean of two positive numbers. Suppose that $AC = x$ and $BC = y$.



 a. Find the length of the radius of circle O in terms of x and y.
 b. Find CD in terms of x and y. Recall from geometry the Secant Length Theorem: $AC \cdot CB = CD \cdot CE$. In this situation, $CE = CD$.
 c. In $\triangle OCD$, which side corresponds to the arithmetic mean of x and y?
 d. In $\triangle OCD$, which side corresponds to the geometric mean of x and y?
 e. Which of the segments you named in Parts c and d must be longer, and why?
 f. In what situation can the two means be equal?

QY ANSWERS

1. $4a^2b^5$
2. $w^3\sqrt[5]{w^2}$