Lesson 8-4 Radical Notation for *n*th Roots

Vocabulary

radical sign, $\sqrt{}$

$\sqrt[n]{x}$ when $x \geq 0$

geometric mean

BIG IDEA For any integer *n*, the largest real *n*th root of *x* can be represented either by $x^{\frac{1}{n}}$ or by $\sqrt[n]{x}$.

As you have learned, the positive *n*th root of a positive number *x* can be written as a power of *x*, namely as $x^{\frac{1}{n}}$. This notation allows all the properties of powers to be used with these *n*th roots. Another notation, using the **radical sign** $\sqrt{}$, can be used to represent all of the positive *n*th roots and some other numbers. However, it is more difficult to see the properties of powers with this notation.

Recall that when *x* is positive, $\sqrt{x}$ stands for the positive square root of *x*. Because $x^{\frac{1}{2}}$ also stands for this square root when *x* is positive, $\sqrt{x} = x^{\frac{1}{2}}$. Similarly the *n*th root of a positive number can be written in two ways.

Definition of $\sqrt[n]{x}$ when $x \geq 0$

When *x* is nonnegative and *n* is an integer ≥ 2, then $\sqrt[n]{x} = x^{\frac{1}{n}}$.

Thus, when *x* is positive, $\sqrt[n]{x}$ is the positive *n*th root of *x*. When $n = 2$, we do not write $\sqrt[2]{x}$, but use the more familiar symbol $\sqrt{x}$.

STOP **QY1**

$\sqrt[n]{x}$ is used to calculate a type of average called a *geometric mean*.

The Geometric Mean

You know several ways to describe a set of values with a single statistic such as an average test score or a median housing price. Such values are called *measures of center,* or *measures of central tendency.*

Suppose a data set has *n* values. If you add the values and divide by *n,* you have calculated the *arithmetic mean* of the set. This is commonly called *the average* of the set, but there are other averages. If, instead of adding, you *multiply* the numbers in the set, and instead of dividing you *take the nth root* of the product of the items in the list, you obtain the **geometric mean**.

Mental Math

Vanessa runs an apple orchard. She sells apples for $1.50 per pound, rounded to the nearest half pound.

a. How much does she charge for 3.2 pounds of apples?

b. How much does she charge for 4.6 pounds of apples?

c. Suppose one apple weighs about a third of a pound. About how many apples can you buy for $6?

d. About how many apples can you buy for $10?

QY1

Solve for *x*:
$\sqrt[7]{78{,}125} = 78{,}125^x$.

The geometric mean may be used when numbers are quite dispersed, to keep one very large number from disproportionately affecting the measure of center. For this reason, the geometric mean is the standard measure of center for data about pollutants and contaminants. The geometric mean is also used to compute an overall rate of percent increase or decrease, as you will see in the Questions.

Example 1

In a water-quality survey, 8 samples give the following levels of *E. coli* bacteria.

Sample	1	2	3	4	5	6	7	8
Count (per 100 mL)	2	54	145	38	597	1152	344	87

Use the geometric mean to compute an average level of bacteria.

Solution Because there are eight numbers, the geometric mean is the 8th root of their product.

$$\sqrt[8]{2 \cdot 54 \cdot 145 \cdot 38 \cdot 597 \cdot 1152 \cdot 344 \cdot 87} \approx 102.6.$$

A typical sample has about 103 bacteria per 100 mL.

STOP QY2

QY2

A ninth sample is taken with an *E. coli* count of 436 per 100 mL. Estimate the average number of bacteria across all nine samples.

Which *n*th Root Does $\sqrt[n]{x}$ Represent?

The symbol $\sqrt[n]{x}$, like $x^{\frac{1}{n}}$, does not represent all nth roots of x. When x is positive and n is even, x has two real nth roots, but only the *positive* real root is denoted by $\sqrt[n]{x}$. Thus, although 3, -3, $3i$, and $-3i$ are all fourth roots of 81, $\sqrt[4]{81} = 81^{\frac{1}{4}} = 3$ only. The negative real fourth root can be written $-81^{\frac{1}{4}}$ or $-\sqrt[4]{81}$, both of which equal -3. Note that the four complex solutions to $x^4 = 81$ are $\pm\sqrt[4]{81}$ and $\pm\sqrt[4]{-81}$, that is, ±3 and $\pm3i$.

GUIDED

Example 2

a. Set a CAS to real-number mode. Evaluate $\sqrt[4]{6561}$ and explain its meaning.

b. Set a CAS to complex-number mode. Find all fourth roots of 6561.

Solution

a. $\sqrt[4]{6561} = \underline{\ ?\ }^{\frac{1}{4}}$ represents the (positive/negative) number whose $\underline{\ ?\ }$ power is 6561. $(6561)^{\frac{1}{4}} = \underline{\ ?\ }$ because $\underline{\ ?\ }^4 = 6561$. So, $\sqrt[4]{6561} = \underline{\ ?\ }$.

b. On a CAS, solve the equation $\underline{\ ?\ }$ in complex-number mode. The solutions are $\underline{\ ?\ }$, $\underline{\ ?\ }$, $\underline{\ ?\ }$, $\underline{\ ?\ }$. These solutions are all the $\underline{\ ?\ }$ roots of 6561.

An early form of the $\sqrt{\ }$ symbol (which looked like a check mark) first appeared in the 1500s, and René Descartes modified it in the early 1600s into the form we use today. The origin of the word *radical* is the Latin word *radix*, which means "root."

Albert Girard suggested the symbol $\sqrt[n]{\ }$, and it first appeared in print in 1690. As you know from Chapter 7, most graphing calculators let you enter the rational power form `x^(1/n)` to find $\sqrt[n]{x} = x^{\frac{1}{n}}$. Some graphing calculators also let you enter radical forms of roots as shown in Example 2.

Radicals for Roots of Powers

Because radicals are powers, all properties of powers listed in Chapter 7 apply to radicals. In particular, because $\sqrt[n]{x} = x^{\frac{1}{n}}$ for $x > 0$, the mth powers of these numbers are equal. That is, $(\sqrt[n]{x})^m = \left(x^{\frac{1}{n}}\right)^m$, which equals $x^{\frac{m}{n}}$. If x is replaced by x^m in the definition of $\sqrt[n]{x}$, the result is $\sqrt[n]{x^m} = (x^m)^{\frac{1}{n}}$, which also equals $x^{\frac{m}{n}}$. Thus, there are two radical expressions equal to $x^{\frac{m}{n}}$.

Root of a Power Theorem

For all positive integers $m \geq 2$ and $n \geq 2$,

$$\sqrt[n]{x^m} = (\sqrt[n]{x})^m = x^{\frac{m}{n}} \text{ when } x \geq 0.$$

STOP **QY3**

Notice that $x \geq 0$ in both the Root of a Power Theorem and QY3. This is because the Root of a Power Theorem only applies to positive bases.

STOP **QY4**

QY3

Suppose $x \geq 0$. Use the Root of a Power Theorem to write $\sqrt[6]{x^{18}}$ in two different ways.

QY4

Show that $\sqrt[6]{x^{18}} \neq x^3$ when $x = -2$.

Roots of Roots

Consider the sequence 625, 25, 5, $\sqrt{5}$, ... in which each number is the square root of the preceding number. You can define this sequence recursively.

$$\begin{cases} s_1 = 625 \\ s_n = \sqrt{s_{n-1}}, \text{ for integers } n \geq 2 \end{cases}$$

So, $s_2 = \sqrt{625} = 25$

$$s_3 = \sqrt{\sqrt{625}} = \sqrt{25} = 5$$

$$s_4 = \sqrt{\sqrt{\sqrt{625}}} = \sqrt{\sqrt{25}} = \sqrt{5} \approx 2.24$$

STOP **QY5**

Rewriting the radicals as rational exponents provides a way to deal with roots of roots.

Example 3

Rewrite $\sqrt{\sqrt{\sqrt{6}}}$ using rational exponents. Is this expression an *n*th root of 6? Justify your answer.

Solution $\sqrt{\sqrt{\sqrt{6}}} = \left(\left(6^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = \left(6^{\frac{1}{2}}\right)^{\frac{1}{4}} = 6^{\frac{1}{8}}$ by the Power of a Power Postulate. So, by the definition of nth root, $\sqrt{\sqrt{\sqrt{6}}}$ is the positive 8th root of 6.

Check Enter $\sqrt{\sqrt{\sqrt{6}}}$ on a calculator to see that $\sqrt{\sqrt{\sqrt{6}}} = 6^{\frac{1}{8}}$.

In general, when $x > 0$, it is more common to write $x^{\frac{1}{8}}$ as the single radical $\sqrt[8]{x}$, rather than as $\sqrt{\sqrt{\sqrt{x}}}$. But if you only have a square root key on your calculator, then it is nice to know you can calculate 8th roots.

QY5

Is this a geometric sequence? Why or why not?

Questions

COVERING THE IDEAS

1. Who is credited with first using the radical symbol in its current form?

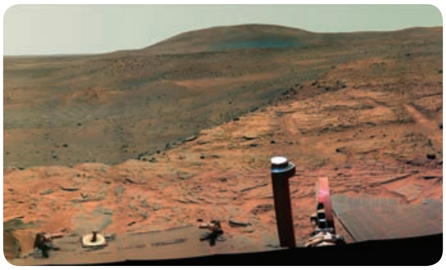
2. Who is credited with first using the symbol $\sqrt[n]{}$?

3. **True or False** $\sqrt[n]{x} = (x)^{\frac{1}{n}}$ for all x.

4. a. How is the geometric mean of n numbers calculated?
   b. Identify a situation in which the geometric mean is the preferred measure of center.

5. The table at the right gives the masses of the eight major planets as a ratio with Earth's mass. Find the geometric mean of these masses.

Planet	Planet's Mass / Earth's Mass
Mercury	0.06
Venus	0.82
Earth	1
Mars	0.11
Jupiter	318
Saturn	95
Uranus	14.5
Neptune	17.2



The Mars rover Spirit took this photo from the eastern edge of the plateau called "Home Plate."

6. Evaluate without a calculator.
   a. $\sqrt{64}$
   b. $\sqrt[3]{64}$
   c. $\sqrt[6]{64}$
   d. $\sqrt[10]{64^{10}}$

In 7 and 8, use a calculator to approximate to the nearest hundredth.

7. $\sqrt[17]{7845}$

8. $\sqrt[8]{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}$

9. **Fill in the Blank** Complete the following statement of the Root of a Power Theorem. For all positive integers m and n with $m \geq 2$ and $n \geq 2$, when $x \geq 0$, $x^{\frac{m}{n}} = \underline{\ ?\ }$.

10. Refer to Example 2. Write all the complex fourth roots of 625.

11. a. Find all the complex fourth roots of 14,641.
    b. Which root in Part a is $\sqrt[4]{14{,}641}$?

In 12–14, write as a single power using a rational exponent. Assume all variables are positive.

12. $\sqrt[6]{z^{10}}$ 13. $\sqrt[5]{c^{15}}$ 14. $\left(\sqrt[14]{t}\right)^7$

15. Rewrite $\sqrt{\sqrt{\sqrt{x}}}$ with a rational exponent, for $x \geq 0$.

16. Rewrite $\sqrt{\sqrt{\sqrt{81}}}$ using a single radical sign.

APPLYING THE MATHEMATICS

Multiple Choice In 17 and 18, which of the expressions is not equivalent to the other two?

17. A $6^{\frac{12}{36}}$ B $\sqrt[12]{6^2}$ C $\sqrt[12]{36}$

18. A $p^{\frac{3}{2}}$ B $\left(\sqrt[4]{p^2}\right)^6$ C $\left(\sqrt{\sqrt{p}}\right)^6$

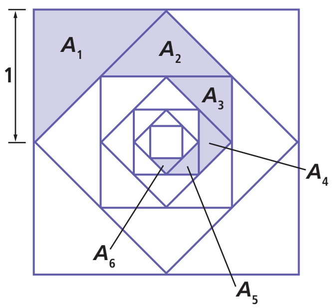
19. From the data on page 514, you can compute that the unadjusted median home value in the United States in 2000 was more than 40 times the median value in 1940. The decade-to-decade percentage increase in housing values is summarized in the table below.

Years	1940–1950	1950–1960	1960–1970	1970–1980	1980–1990	1990–2000
% Increase (rounded)	150	62	43	178	68	51
Size-change factor	2.50	?	?	?	?	?

a. Fill in the table by converting each percent into a size-change factor.

b. Compute the geometric mean of the size-change factors you found in Part a, and determine the average percentage increase per decade in home values from 1940 to 2000.

20. Recall from Chapter 7 the Baravelle spiral, in which squares are created by connecting midpoints of sides of larger squares. In the Baravelle spiral shown below, the leg length of right triangle A_1 is 1 unit.



a. What are the lengths of the hypotenuses of right triangles A_1, A_2, and A_3?

b. The lengths of the hypotenuses $L_1, L_2, L_3, \ldots$ are a geometric sequence. Write an explicit formula for L_n, the sequence of hypotenuse lengths in this Baravelle spiral.

21. A sphere has volume V in^3, where $V = \frac{4}{3}\pi r^3$. Express the length of the radius in terms of the volume
   a. using radical notation.
   b. using a rational exponent.

22. Consider the formula $m = 1.23x^3b$ where m, x, and b are all positive. Solve the formula for x using radical notation.

23. Use the expression $\sqrt[4]{\sqrt[4]{43{,}046{,}721z^{48}}}$.
   a. Rewrite the expression with a rational exponent.
   b. Rewrite the expression as one radical.
   c. Evaluate the expression when $z = 2$.

In 24 and 25, write each expression in simplest radical form using no fraction exponents. Assume all variables are positive.

24. $\sqrt{\sqrt{y^{\frac{1}{4}}}}$

25. $\dfrac{\sqrt{k^{\frac{1}{2}}}}{\sqrt{\sqrt{k}}}$

REVIEW

26. Let $h(x) = x^2$ and $k(x) = x^{-\frac{1}{2}}$. (Lessons 8-3, 8-1, 7-8, 7-2)
   a. Find $h \circ k(x)$.
   b. Are h and k inverses of each other? How can you tell?

27. The height of a baseball that has been hit is described by the parabola $h(x) = -0.00132x^2 + 0.545x + 4$, where x is the distance in feet from home plate. The effect of a 15-mph wind in the direction the ball is traveling is given by $w(x) = 0.9x$. (Lessons 8-1, 6-4)
   a. Which of $w(h(x))$ or $h(w(x))$ represents this situation? (*Hint:* The wind is blowing before the ball is pitched and hit.)
   b. Write a formula for your answer to Part a.
   c. How far from home plate will the ball land if there is no wind?
   d. How far from home plate will the ball land if there is a 15-mph wind in the direction of travel?

In 28 and 29, write without an exponent. Do not use a calculator. (Lesson 7-8)

28. $9^{-\frac{3}{2}}$

29. $-\frac{1}{5}^{-3}$

30. Write the reciprocal of $3 + 2i$ in $a + bi$ form. **(Lesson 6-9)**

31. If $g(t) = \frac{\frac{t-4}{t+3}}{\frac{t-2}{t+1}}$, what values of t are not in the domain of g? **(Lesson 1-4)**

32. A math text from 1887 gives the following apothecary weights: 3 scruples = 1 dram, 8 drams = 1 ounce. How many scruples are in two and a half ounces? **(Previous Course)**

In colonial times, apothecaries provided medical treatment, prescribed medicine, and even performed surgery.

EXPLORATION

33. In Parts a–c, use a calculator to estimate to the nearest hundredth.

a. $\sqrt{1 + \sqrt{1 + \sqrt{1}}}$

b. $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}$

c. $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}}$

d. Use a CAS to solve the equation $x = \sqrt{1 + x}$ such that $x \geq 0$.

e. Approximate your solution in Part d to the nearest hundredth, and compare it to your answers in Parts a, b, and c. What do you think is happening?

QY ANSWERS

1. $x = \frac{1}{7}$

2. about 120 bacteria per 100 mL

3. $\left(\sqrt[6]{x}\right)^{18}$ and x^3

4. When $x = -2$, $\sqrt[6]{x^{18}} = \sqrt[6]{(-2)^{18}} = \sqrt[6]{2^{18}}$, a positive number, but $x^3 = -8$, a negative number.

5. No; the ratio between consecutive terms is not constant.