Lesson

8-3 Properties of Inverse Functions

Vocabulary

inverse function, f^{-1}

▶ **BIG IDEA** When a function has an inverse, the composite of the function and its inverse in either order is the identity function.

Using an Inverse Function to Decode a Message

In Chapter 5, you used matrices to code and decode messages. You can use functions to do this too. For instance, suppose you and your friend have agreed on a key that pairs numbers with the letters they stand for and an encoding function $f(x) = 3x + 11$. This means that your friend began with a number code for a letter (the input), multiplied that number by 3, and then added 11 to get the output. For example, if $A = 65$ it is encoded as $3 \cdot 65 + 11 = 206$. Suppose your friend now sends you this coded message:

212 248 215 218 260 107 206 257 218 107 212 248 248 239

To decode the message, you "undo" your friend's coding; subtract 11 and divide the result by 3 to get the original input: $\frac{206 - 11}{3} = 65$. This "undoing" function is the **inverse function** of f. In Question 13, you will use this inverse function to decode the above message.

Mental Math

Give an equation for the graph of the parabola $y = -2x^2$ under the given translation.

a. $T_{1,3}$

b. $T_{-2,-2}$

c. $T_{0.3,5.4}$

Formulas for Inverses Using $f(x)$ Notation

There are many ways to obtain a formula for the decoding (inverse) function described above. Here is one way.

Start with the original function equation.	$f(x) = 3x + 11$
Replace $f(x)$ with y.	$y = 3x + 11$
Use the Inverse-Relation Theorem and switch x and y in the equation for f. This represents the decoding function in which y is the input from your friend and x is the output.	$x = 3y + 11$
Solve for y. For a given output x from your friend's encoding function, you can use this function to calculate the original input y.	$y = \frac{x - 11}{3}$
Substitute $g(x)$ for y. This describes the inverse, g.	$g(x) = \frac{x - 11}{3}$

Composites of a Function and Its Inverse

Activity 1

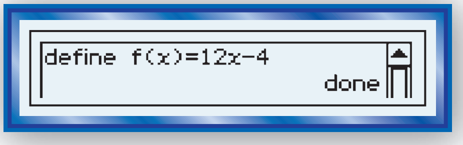
MATERIALS CAS

Work with a partner to complete the following steps. One of you should work with $f(x) = 12x - 4$, the other with $f(x) = \frac{1}{3} + \frac{x}{12}$.

Step 1 Using the method described on the previous page, find the inverse of your function. Call your inverse g and write a formula for it.

Step 2 On your CAS, define your functions f and g. Evaluate $f(g(x))$ and $g(f(x))$.

Step 3 Compare your results with those of your partner.



Your results for $f(g(x))$ and $g(f(x))$ in Activity 1 hold for any functions f and g that are inverses. The converse is also true. Both are summarized in the following theorem.

Inverse Functions Theorem

Two functions f and g are inverse functions if and only if:

(1) For all x in the domain of f, $g(f(x)) = x$, and

(2) for all x in the domain of g, $f(g(x)) = x$.

Proof We must prove a statement and its converse. So the proof has two parts.

(I) *The "only if" part:* Suppose f and g are inverse functions. Let (a, b) be any ordered pair in the function f. Then $f(a) = b$. By the definition of inverse, the ordered pairs in g are the reverse of those in f, so (b, a) is an ordered pair in the function g. Thus $g(b) = a$. Now take the composites.

For any number a in the domain of f: $g(f(a)) = g(b) = a$.

For any number b in the domain of g: $f(g(b)) = f(a) = b$.

(II) *The "if" part:* Suppose (1) and (2) in the statement of the theorem are true. Again let (a, b) be any point on the function f. Then $f(a) = b$, and so $g(f(a)) = g(b)$. But using (1), $g(f(a)) = a$, so, by transitivity, $g(b) = a$. This means that (b, a) is in the function g. So g contains all the points obtained by reversing the coordinates in f.

By the same reasoning we can show that f contains all points obtained by reversing the coordinates of g. Thus, f and g are inverse functions.

STOP **QY1**

▶ QY1

If $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}x - 6$, are f and g inverses of each other?

Notation for Inverse Functions

Recall that an *identity function* is a function that maps each object in its domain onto itself. Another way of stating the Inverse Functions Theorem is to say that the composite of two functions that are inverses of each other is the identity function I with equation $I(x) = x$.

When an operation on two elements of a set yields an identity element for that operation, then we call the elements *inverses*. For example, 2 and $\frac{1}{2}$ are multiplicative inverses because $2 \cdot \frac{1}{2} = 1$, the identity element for multiplication. This is the reason that we call g the inverse of f, and f the inverse of g: $f(g(x)) = g(f(x)) = I(x)$, the identity function.

The multiplicative inverse of a number x is designated by x^{-1}. Similarly, when a function f has an inverse, we designate the **inverse function** by the symbol $\mathbf{f^{-1}}$, read "f inverse." For instance, for the functions in Activity 1 you showed that for all x, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$, which is read "$f$ of f inverse of x equals f inverse of f of x equals x."

GUIDED

Example

Let $h : x \rightarrow \frac{8(x-2)+5}{3}$. Find a rule for h^{-1}.

Solution Use a process like that shown for the function f at the beginning of the lesson. From the given information,

$$h(x) = \frac{8(x-2)+5}{3}.$$

Simplify. $\quad = \frac{? - ?}{3}$

Substitute y for $h(x)$.

$$y = \underline{\quad ? \quad}$$

An equation for the inverse is found by switching x and y.

$$x = \underline{\quad ? \quad}$$

Solve this equation for y.

$$3x = \underline{\ ?\ } - \underline{\ ?\ }$$

$$3x + \underline{\ ?\ } = \underline{\ ?\ }$$

$$y = \underline{\quad ? \quad}$$

So $\quad h^{-1}(x) = \underline{\quad ? \quad}$.

STOP **QY2**

QY2

"Halving" is an inverse for the "doubling" function. What is an inverse for the squaring function?

In earlier lessons, you worked with the functions with equations $y = x^n$, "taking the nth power," and $y = x^{\frac{1}{n}}$, "taking the nth root." For all $x \geq 0$, it is reasonably easy to show that these functions are inverse functions.

Power Function Inverse Theorem

If $f(x) = x^n$ and $g(x) = x^{\frac{1}{n}}$ and the domains of f and g are the set of *nonnegative* real numbers, then f and g are inverse functions.

Proof First, show that $f \circ g(x) = x$ for all x in the domain of g.

Substitute. $f \circ g(x) = f\left(x^{\frac{1}{n}}\right)$ Definition of g

Because x is a nonnegative number, $x^{\frac{1}{n}}$ is always defined and you can apply f.

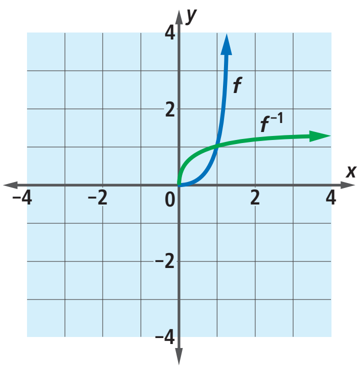
$= \left(x^{\frac{1}{n}}\right)^n$ Definition of f

$= x^1 = x$ Power of a Power Postulate

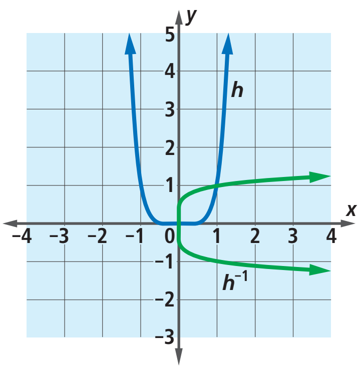
Now you need to show that $g \circ f(x) = x$ for all x in the domain of f. You are asked to do this in Question 12.

An instance of the Power Function Inverse Theorem is illustrated by the graphs of $f(x) = x^6$ and $f^{-1}(x) = x^{\frac{1}{6}}$ at the left below. The graphs are reflection images of each other over the line with equation $y = x$. Notice also that the domain of these functions is the set of nonnegative real numbers.

The function $h(x) = x^6$ with domain the set of *all* real numbers is graphed at the right below. Its inverse is not a function. Notice that the graph of h does not pass the horizontal-line test.



$f(x) = x^6$
Domain of f:
set of nonnegative real numbers
$f^{-1}(x) = x^{\frac{1}{6}}$
Domain of f^{-1}:
set of nonnegative real numbers



$h(x) = x^6$
Domain of h:
set of real numbers

The inverse of h
is not a function.

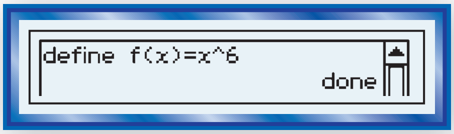
A CAS can help you find equations for inverse relations and to limit domains of functions so that their inverses are also functions.

Activity 2

MATERIALS CAS

Use a CAS to find an equation for the inverse of $y = f(x) = x^6$.

Step 1 Clear all variables in a CAS. Define $f(x) = x^6$.



Step 2 Switch x and y and solve $x = f(y)$ for y.

Step 3 Your display should show an equation for the inverse of f. Is this inverse a function? If not, on what domain of f is f^{-1} a function?

Questions

COVERING THE IDEAS

1. Suppose f is a function. What does the symbol f^{-1} represent, and how is it read?

2. Refer to Activity 1. Show that $f(g(2)) = g(f(2))$.

3. Let $f(x) = 5 - 2x$ and $g(x) = \frac{x - 5}{-2}$. Are f and g inverses of each other? Justify your answer.

4. The function $h : x \rightarrow 5x + 7$ has an inverse h^{-1} which is a function.
   a. Find a formula for $h^{-1}(x)$.
   b. Check your answer to Part a by finding $h \circ h^{-1}(x)$ and $h^{-1} \circ h(x)$.

5. Consider the "adding 15" function. Call it A.
   a. What is a formula for $A(x)$?
   b. Give a formula for $A^{-1}(x)$.
   c. What is an appropriate name for A^{-1}?
   d. What is $A^{-1} \circ A(63)$?

In 6 and 7, an equation for a function f is given. The domain of f is the set of real numbers. Is the inverse of f a function? If it is, find a rule for f^{-1}. If not, explain why not.

6. $f(x) = 21x$

7. $f(x) = 21$

8. **Fill in the Blank** For any function f that has an inverse, if x is in the range of f, then $f \circ f^{-1}(x) = \underline{\ ?\ }$.

9. If $f(x) = x^7$ with domain the set of nonnegative real numbers, give a formula for $f^{-1}(x)$.

10. Let $m(x) = \frac{x}{5}$. This function could be named "dividing by 5."
    a. Find an equation for m^{-1}.
    b. Give an appropriate name to m^{-1}.

11. a. Use a CAS to find an equation for the inverse of $f(x) = -\frac{(x + 12)^3}{2}$.
    b. Is the inverse of f a function? If so, what is its domain? If not, over what domain of f is the inverse a function?

12. Prove the second part of the Power Function Inverse Theorem in this lesson.

13. Recall the code your friend sent you at the beginning of the lesson. Use the following table to decode the message. A space is represented by 32.

Code	65	66	67	68	69	70	71	72	73	74	75	76	77
Letter	A	B	C	D	E	F	G	H	I	J	K	L	M
Code	78	79	80	81	82	83	84	85	86	87	88	89	90
Letter	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

APPLYING THE MATHEMATICS

14. The function S with $S(n) = (n - 2) \cdot 180°$ maps the number of sides of a convex polygon onto the sum of the measures of its interior angles.
    a. Find a formula for $S^{-1}(n)$.
    b. Use the formula to determine the number of sides of a polygon when the sum of the interior angle measures of the polygon is 3780.

15. Let $g(x) = x^{\frac{2}{3}}$ and $h(x) = x^{\frac{3}{2}}$ with $x \geq 0$.
    a. Graph g and h on the same set of axes.
    b. Describe how the graphs are related.
    c. Find $h(g(x))$ and $g(h(x))$.
    d. **Fill in the Blank** The answer to Part c means that since $64^{\frac{3}{2}} = 512$, $\underline{\ \ \ ?\ \ \ }$.
    e. Are g and h inverses? Why or why not?

In 16–18, find an equation for the inverse of the function with the given equation in the form $y =$ ___?___.

16. $y = \frac{1}{x}$ **17.** $y = \frac{1}{2}x + 7$ **18.** $y = x^3 - 2$

19. If f is any function that has an inverse, what is $(f^{-1})^{-1}$? Explain your answer in your own words.

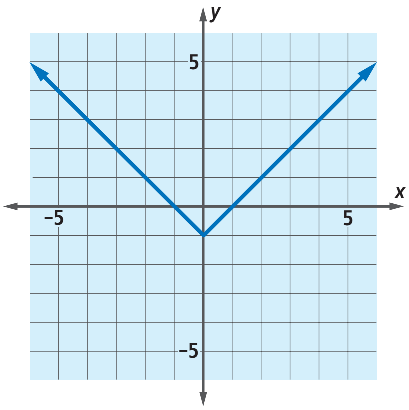
20. **a.** Explain why the inverse of $g(x) = x^2$ is not a function when the domain of g is the set of all real numbers.

b. Split the graph of $g(x) = x^2$ along its line of symmetry. For each half,

i. State the domain and the corresponding range.

ii. Give a formula for the inverse of that half.

21. **Multiple Choice** Consider the absolute-value function $y = |x| - 1$ graphed below. Which of the following domains gives a function whose inverse is also a function?



A $\{x \mid -2 \le x \le 5\}$ **B** $\{x \mid x \ge 1\}$ **C** $\{x \mid x \ge -4.5\}$

REVIEW

22. Sandy makes \$32,000 per year plus \$24 per hour for each hour over 2000 hours that she works per year. An equation for her total income $I(h)$ when she works h hours is $I(h) = 32{,}000 + 24(h - 2000)$, for $h > 2000$. **(Lesson 8-2)**

a. Find a rule for the inverse of I.

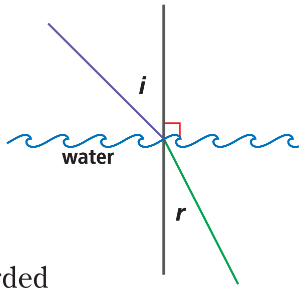
b. What does the inverse represent?

23. **Fill in the Blanks** Suppose $f : x \to 2x + 5$ and $g : x \to x^{-2}$. **(Lessons 8-2, 8-1)**

a. $f \circ g(-2) =$ ___?___ **b.** $g \circ f(-2) =$ ___?___

c. $f(g(x)) =$ ___?___ **d.** $g(f(x)) =$ ___?___

24. When a beam of light in air strikes the surface of water it is *refracted*, or bent. Below are the earliest known data (translated into modern units) on the relation between i, the measure of the angle of incidence in degrees, and r, the measure of the angle of refraction in degrees. The measurements are recorded in *Optics* by Ptolemy, a Greek scientist who lived in the second century BCE. **(Lessons 6-6, 3-5)**



i	10°	20°	30°	40°	50°	60°	70°	80°
r	8°	15.5°	22.5°	29°	35°	40.5°	44.5°	50°

a. Draw a scatterplot of these data.
b. Fit a quadratic model to these data.
c. Fit a linear model to these data.
d. Which model seems more appropriate? Explain your decision.

25. Line ℓ is parallel to the line with equation $y = \frac{1}{4}x + 7$. **(Lesson 3-1)**
a. What is the slope of ℓ?
b. What numbers are possible y-intercepts of ℓ?

EXPLORATION

26. The *Shoe and Sock Theorem*: Let g be the function with the rule "put your sock on the input." Let f be the function with the rule "put your shoe on the input."
a. Explain in words how to find the output of $f \circ g$ (your foot).
b. What are the rules for g^{-1} and f^{-1}?
c. What is the result of applying $f \circ g$ followed by $(f \circ g)^{-1}$ to your foot?
d. What two steps must be taken to apply $(f \circ g)^{-1}$ to your foot? In what order must the steps be taken?
e. Translate your answer to Part d into symbols: $(f \circ g)^{-1} =$ ___?___.
f. Now let $f(x) = x^5$ and $g(x) = 7x - 19$. Find formulas for $f^{-1}(x)$ and $g^{-1}(x)$ and use them to find a formula for $(f \circ g)^{-1}(x)$.

QY ANSWERS

1. No; $f(g(x)) = f\left(\frac{1}{2}x - 6\right) = 2\left(\frac{1}{2}x - 6\right) + 6 = x - 12 + 6 = x - 6 \neq x$

2. the square root function