Lesson

8-2 Inverses of Relations

Vocabulary

inverse of a relation

horizontal-line test

BIG IDEA Every relation has an *inverse*. Some of these inverses are functions.

In 1929, astronomers Edwin Hubble and Milton Humason announced their discovery that other galaxies are moving away or *receding* from our galaxy, the Milky Way. Galaxies that are farther away appear to be receding faster. The relationship between distance and recession speed is given in the table below. Distance is given in megaparsecs (Mpc), with 1 Mpc ≈ 3.26 million light years. Speed is given in kilometers per second.

x = Distance (Mpc)	y = Speed $\left(\frac{\text{km}}{\text{sec}}\right)$
0	0
5	350
10	700
15	1050
20	1400

In this table, the left column has values of the domain variable x and the right column has values of the range variable y. The function that maps distance onto recession speed is called *Hubble's Law* and is described by the equation $y = 70x$.

Astronomers use Hubble's Law to estimate the distance to a galaxy based on its apparent recession speed, which is measured by shifts in the spectrum of light from the galaxy. In other words, the speed becomes the domain variable x and the distance becomes the range variable y. The function mapping the speed onto the distance can be described by the equation $x = 70y$. Solving this equation for y, $y = \frac{x}{70}$.

x = Speed $\left(\frac{\text{km}}{\text{sec}}\right)$	y = Distance (Mpc)
0	0
350	5
700	10
1050	15
1400	20

Mental Math

Nina is pulling chips one at a time out of a bag that contains 4 red chips, 5 black chips, and 1 blue chip. She replaces the chip after each pull. What is the probability that she pulls

a. the blue chip twice in a row?

b. a red chip, then a black chip?

c. a red chip n times, then a black chip m times?

Galaxies NGC 2207 and IC 2163 are 140 million light years away from us in the direction of the Canis Major constellation. They are expected to meld in 500 million years.

The relations with equations $y = 70x$ and $y = \frac{1}{70}x$ are related. The ordered pairs of each one are found by switching the values of x and y in the other. Two relations that have this property are called *inverse relations*.

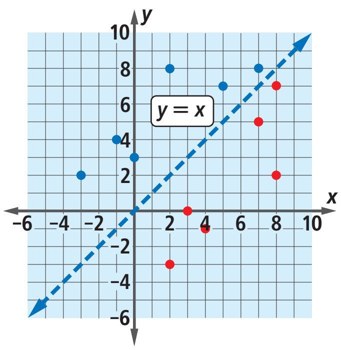
Definition of Inverse of a Relation

The **inverse of a relation** is the relation obtained by switching the coordinates of each ordered pair in the relation.

Example 1

Let $f = \{(-3, 2), (-1, 4), (0, 3), (2, 8), (5, 7), (7, 8)\}$. Find the inverse of f.

Solution Switch the coordinates of each ordered pair. Call the inverse g. Then $g = \{(2, -3), (4, -1), (3, 0), (8, 2), (7, 5), (8, 7)\}$.



The blue dots at the right are a graph of the function f from Example 1. The red dots are a graph of the inverse g.

Recall that the points (x, y) and (y, x) are reflection images of each other over the line with equation $y = x$. That is, reflection over this line switches the coordinates of the ordered pairs. The graphs of any relation and its inverse are reflection images of each other over the line $y = x$.

The Domain and Range of a Relation and Its Inverse

Recall that the domain of a relation is the set of possible values for the first coordinate, and the range is the set of possible values for the second coordinate. Because the inverse is found by switching the first and second coordinates, the domain and range of the inverse of a relation are the range and domain, respectively, of the original relation. For instance, in Example 1,

$$\text{domain of } f = \text{range of } g = \{-3, -1, 0, 2, 5, 7\},$$

$$\text{range of } f = \text{domain of } g = \{2, 4, 3, 8, 7\}.$$

The theorem on the next page summarizes the ideas you have read so far in this lesson.

Inverse-Relation Theorem

Suppose f is a relation and g is the inverse of f. Then:

1. If a rule for f exists, a rule for g can be found by switching x and y in the rule for f.
2. The graph of g is the reflection image of the graph of f over the line with equation $y = x$.
3. The domain of g is the range of f, and the range of g is the domain of f.

Caution! The word *inverse*, when used in the term *inverse of a relation*, is different than its use in the phrase *inverse variation*.

Determining Whether the Inverse of a Function Is a Function

The inverse of a relation is a relation. But the inverse of a function is not always a function. In Example 1, g is not a function because it contains the two pairs (8, 2) and (8, 7).

GUIDED

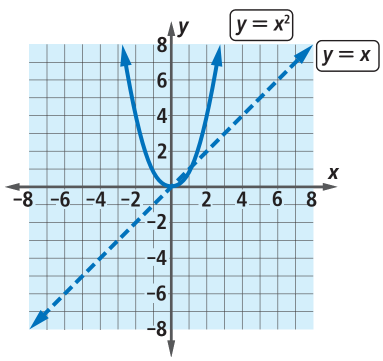
Example 2

Consider the function with domain the set of all real numbers and equation $y = x^2$.

a. What is an equation for the inverse of this function?

b. Graph the function and its inverse on the same coordinate axes.

c. Is the inverse a function? Why or why not?



Solution

a. To find an equation for the inverse, switch x and y in the rule for the function. The inverse of the function with equation $y = x^2$ has equation __?__ = __?__.

b. The graph of __?__ is the parabola at the right. The graph of its inverse is the reflection image of the parabola over the line with equation __?__. Copy the graph and add a graph of the inverse to it.

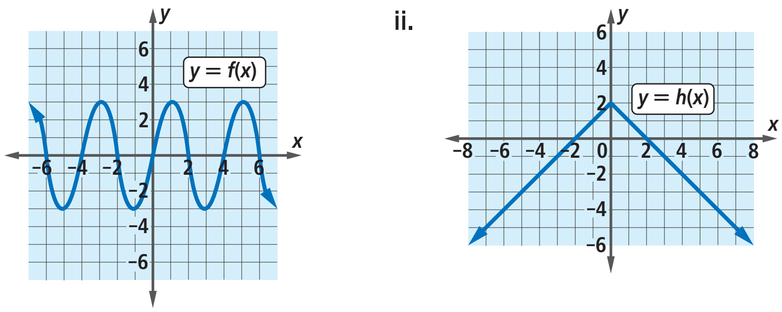
c. Graphs of functions in rectangular coordinates do not include two points with the same first coordinate. The inverse (is/is not) a function because __?__.

In a function, no two points have the same first coordinate. So, if the inverse of a function is a function, no two points of the original function can have the same *second* coordinate. This idea implies that if any horizontal line intersects the graph of a function on a rectangular grid at more than one point, then the function's inverse is not a function. This is called the **horizontal-line test** to check whether the inverse of a function is a function. Notice that the graphs of the functions in Examples 1 and 2 do not pass the horizontal-line test.

Example 3

Two functions are graphed.

i. ii.



a. Tell whether the inverses of the graphed functions are functions. Explain your reasoning.

b. State the domain and range of each function.

c. State the domain and range of each inverse.

Solution

a. Apply the horizontal-line test. For each function, there is a horizontal line that intersects the graph of the function more than once. Neither the inverse of f nor the inverse of h is a function. Both graphs fail the horizontal-line test.

b. Refer to the graphs above. The domain of f is the set of all real numbers. The range is $\{y \mid -3 \le y \le 3\}$. The domain of h is the set of all real numbers. The range is $\{y \mid y \le 2\}$.

c. Use the Inverse-Relation Theorem. The domain of the inverse of f is the range of f, or $\{x \mid -3 \le x \le 3\}$; the range is the domain of f, or the set of all real numbers. The domain of the inverse of h is $\{x \mid x \le 2\}$;the range is the set of all real numbers.

Questions

COVERING THE IDEAS

1. How can the inverse of a relation be found from the coordinates of the points in the relation?

2. Refer to the functions relating distance and speed at the beginning of the lesson.
   a. At what distance away from us is a galaxy moving at roughly 5000 km/sec?
   b. The photograph at the right shows the disk galaxy NGC 5866 as it appears from Earth, edge-on. NGC 5866 is about 44 million light years, or 13.5 megaparsecs, from the Milky Way. How fast is NGC 5866 receding?

This photograph was taken by the Hubble Space Telescope (named in the astronomer's honor) in June, 2006.

In 3 and 4, let $f = \{(-4,-9), (-3, -7), (0, -1), (3, 5), (4, 7), (6, 11)\}$.

3. a. Find the inverse of f.
   b. Graph f and its inverse on the same set of axes.
   c. How are the two graphs related?
   d. Write an equation for the function f.
   e. Write an equation for the inverse of f.

4. Give the elements of each set.
   a. the domain of f
   b. the range of f
   c. the domain of the inverse of f
   d. the range of the inverse of f

5. Explain how the graphs of any relation f, its inverse, and the line with equation $y = x$ are related.

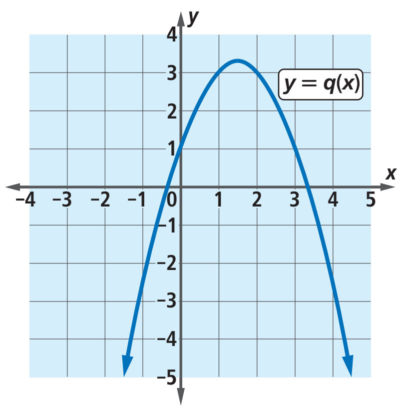
In 6 and 7, give an equation for the inverse of the relation.

6. $y = 3x$

7. $y = \frac{1}{5}x - 2$

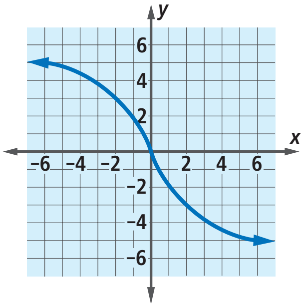
8. Refer to Example 2.
   a. Write an equation for the inverse of $y = x^2$.
   b. Find two points other than those mentioned in the lesson that show the inverse is not a function.

9. The graph of function q is given at the right. Explain if there is a way to tell in advance whether the inverse of q is a function.

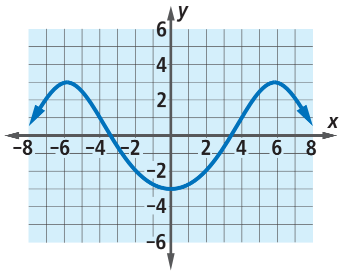


In 10–12, is the inverse of the graphed function a function? Explain.

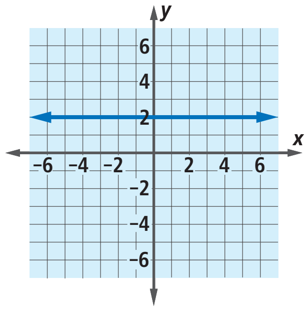
10.



11.



12.



APPLYING THE MATHEMATICS

13. Describe the inverse of the doubling function $D: x \rightarrow 2x$.

14. Consider the function f with equation $f(x) = -3$.

a. Give 5 ordered pairs in f.

b. Find 5 ordered pairs in the inverse of f.

c. Draw graphs of f and its inverse on the same set of axes.

d. What transformation maps the graph of f onto the graph of its inverse?

15. a. Draw the graph of $y = 3x + 7$.

b. Find an equation for its inverse and solve it for y.

c. Graph the inverse on the same set of axes.

d. Is the inverse a function? Explain your answer.

e. How are the slopes of the graphs of the function and its inverse related?

16. a. Graph the inverse of the absolute-value function $y = |x|$.

b. Is the inverse a function? Explain why or why not.

17. In 2008, one U.S. dollar ($) was worth about 0.64 euros (€). An item costing $1 cost €0.64. Let U be the cost of an item in U.S. dollars and let E be the cost of an item in euros.

a. Write an equation for U in terms of E.

b. Write an equation for E in terms of U.

c. Are the two functions you found in Parts a and b inverses of each other? Explain your answer.

REVIEW

18. Let c = the cost of a new car. Then $t(c) = 1.07c$ is the cost after a state sales tax of 7%, and $r(c) = c - 3000$ is the cost after a manufacturer's rebate of \$3000. **(Lesson 8-1)**
    - **a.** Evaluate $t(r(29{,}350))$.
    - **b.** Find a formula for $t(r(c))$.
    - **c.** Explain in words what $t(r(c))$ represents.
    - **d.** Evaluate $r(t(29{,}350))$.
    - **e.** Find a formula for $r(t(c))$.
    - **f.** Explain in words what $r(t(c))$ represents.
    - **g.** If you were the State Treasurer, which would you prefer, $t(r(c))$ or $r(t(c))$, and why?

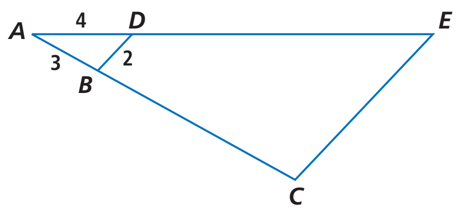
19. Let $f(x) = 5x - 7$, $g(x) = \frac{1}{5}(x + 12)$, and $h(x) = \frac{1}{5}(x + 7)$. **(Lesson 8-1)**
    - **a.** Find $f \circ g(3)$. **b.** Find $g \circ f(3)$. **c.** Find $f \circ h(x)$.
    - **d.** What is another name for the function $f \circ h$?

20. Given $r(x) = 2x^2 - 1$ and $s(x) = x^2 + 3$, find $r \circ s(x)$. **(Lesson 8-1)**

21. **True or False** For all real numbers x, $|-3x| = 3x$. Justify your answer. **(Lesson 6-2)**

22. **True or False** Two matrices A and B are inverses if $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Explain why or why not. **(Lesson 5-5)**

23. In this figure, $\triangle ADB \sim \triangle AEC$. If $DE = 12$, find BC and CE. **(Previous Course)**



EXPLORATION

24. Some functions are their own inverses. One such function has the equation $y = \frac{1}{x}$.
    - **a.** Find at least one more function that is its own inverse.
    - **b.** Graph $y = \frac{1}{x}$ and your answer(s) to Part a. What property do the graphs share?