Lesson 8-1 Composition of Functions

Vocabulary

composite, $g \circ f$

function composition, composition

BIG IDEA The function that results from following one function by another is called the *composite of the two functions*; the operation is called *composition*.

Activity

The Lee Muns Car Dealership offers two incentives to car buyers. They give a 12% discount off the $33,000 sticker price of a new car, as well as a $2000 rebate for an end-of-the-year clearance. If you were given a choice, which incentive would you take first?

Step 1 Take the discount first and then the rebate. Note that a 12% discount means that you pay 88% of the selling price. How much would you pay for the car if you took the discount first?

Step 2 Take the rebate first then apply the discount. How much would you then pay for the car?

Step 3 Which method gives you a lower price for the car? How much money would you save over the other option?

Mental Math

Write in $a + bi$ form.

a. $\frac{6 + 4i}{4}$

b. $5i(2i + i^2)$

c. $(3 + 2i)(3 - 2i)$

d. $(k + ri)(k - ri)$

When the supply of available new cars exceeds customer demand, auto dealers may lower the price to increase demand.

The Composite of Two Functions

Will calculating the discount before the rebate always result in a lower price? Will the difference always be the same? To answer these questions, you can use algebra. Let x be the sticker price, d the discount function, and r the rebate function. If you take the discount first, then the price after the discount will be $d(x)$. If you then take the rebate, you can find the final price by taking the output of the discount function and using it as input to the rebate function. This is given by the expression $r(d(x))$, read "r of d of x."

Example 1

Consider the situation in the Activity. Let $r(x) = x - 2000$ and let $d(x) = 0.88x$.

a. Write an equation for $p(x) = r(d(x))$, that is, the price $p(x)$ if the sticker price is x and discount is taken first, then the rebate.

b. Write an equation for $q(x) = d(r(x))$, that is, the price if the rebate is taken first, then the discount.

c. Is it true for all x that $p(x) < q(x)$?

Solution

a. $p(x) = r(d(x)) = r(0.88x)$ Apply the formula for d.

$= 0.88x - 2000$ Apply the formula for r.

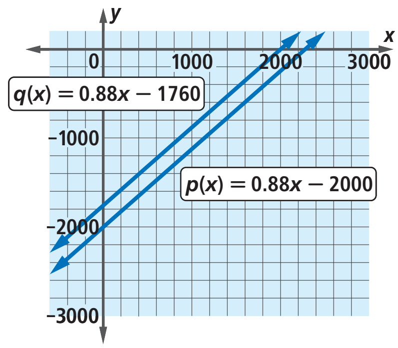
b. $q(x) = d(r(x)) = d(x - 2000)$ Apply the formula for r.

$= 0.88(x - 2000)$ Apply the formula for d.

$= 0.88x - 1760$ Distributive Property

c. Notice that the selling price is $p(x) = r(d(x)) = 0.88x - 2000$ when the discount is given before the rebate, or $q(x) = d(r(x)) = 0.88x - 1760$ when the rebate is given before the discount.

Both $p(x)$ and $q(x)$ are linear equations with the same slope, so the graphs of the functions p and q are parallel lines. The y-intercept of $q(x)$ is \$240 more than the y-intercept of $p(x)$. Hence, $p(x) < q(x)$ for all x. That is, with a 12% discount and a \$2000 rebate, the final selling price is always \$240 less when the discount is taken first, no matter what the sticker price is.



STOP QY1

QY1

Find $r(d(50{,}000))$ and $d(r(50{,}000))$.

Example 1 involves the same ideas you saw in Chapter 4 with transformations. Recall that the result of applying one transformation T after another S is called the composite of the two transformations, written $T \circ S$. Likewise, the function that maps x onto $r(d(x))$ is called the *composite* of the two functions d and r, and is written $r \circ d$.

The composite of two functions is a function. You can describe any function if you know its domain and a rule for obtaining its values. Thus, you define the composite of two functions by indicating its rule and domain.

Definition of Composite

The **composite $g \circ f$** of two functions f and g is the function that maps x onto $g(f(x))$, and whose domain is the set of all values in the domain of f for which $f(x)$ is in the domain of g.

The operation signified by the small circle $\circ$ is called **function composition**, or just **composition**. $g \circ f$ is read "the composite f followed by g" "the composite g following f," or "the composite f then g."

Ways of Writing a Composite

The ways of writing a composite are the same as those used for composites of transformations. Consider the composite $g \circ f$. You can describe the rule for a composite in two ways.

Mapping notation: $g \circ f : x \rightarrow g(f(x))$

$f(x)$ notation: $g \circ f(x) = g(f(x))$

In Example 1, you wrote $r(d(x))$. You could have written this as $r \circ d(x)$.

STOP QY2

> **QY2**
>
> Write "the composite r then d of x" in
>
> **a.** mapping notation.
>
> **b.** $f(x)$ notation.

Is Function Composition Commutative?

From the Activity, $r(d(33{,}000)) = 27{,}040$, and $d(r(33{,}000) = 27{,}280$. This one example is enough to show that *composition of functions is not commutative*. Here is another example.

GUIDED

Example 2

Let $f(x) = x - 2$ and $g(x) = 3x^2$. Calculate

a. $f \circ g(x)$. **b.** $g \circ f(x)$.

Solution

a. $f \circ g$ means first square x and multiply by 3, then subtract 2 from the result.

$f \circ g(x) = f(g(x)) = f(\underline{\ ?\ }) = \underline{\ ?\ } - 2$

b. To evaluate $g \circ f(x)$, first subtract 2 from x, then square the result and multiply by 3.

$g \circ f(x) = g(f(x)) = g(\underline{\ ?\ }) = 3(\underline{\ ?\ })^2 = 3(\underline{\ \ ?\ \ }) = \underline{\ \ ?\ \ }$

Finding the Domain of a Composite of Functions

The domain for a composite is the largest set for which the composition is defined. That is, the domain can include only those values of x for which the first function is defined and that are paired with values that are in the domain of the second function.

Example 3

Let x be a real number. Let $s(x) = \frac{1}{x^2}$, and let f be given by $f(x) = x^2 - 64$. What is the domain of $s \circ f$?

Solution There are no restrictions on the domain of f, but 0 is not in the domain of s. Thus, $f(x)$ cannot be 0. Because $f(x) \neq 0$, $x^2 - 64 \neq 0$; thus, $x^2 \neq 64$. So, x cannot equal 8 or –8. The domain of $s \circ f$ is the set of real numbers other than 8 or –8.

Questions

COVERING THE IDEAS

In 1 and 2, refer to the rebate and discount functions in the Activity.

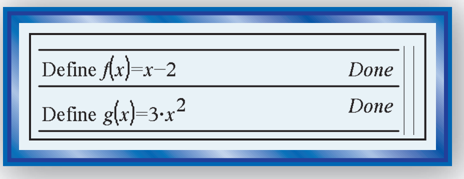
1. a. If the sticker price is \$25,000 and the rebate is calculated first, what is the selling price of the car?
   b. If the sticker price is \$25,000 and the discount is calculated first, what is the selling price of the car?
2. Suppose the car dealer changes the rebate to \$3000 and the discount to 15%.
   a. Which gives a lower selling price, taking the rebate first or taking the discount first? Justify your answer.
   b. Will the difference in selling price between taking the discount first versus taking the rebate first always be the same? Explain why or why not.

In 3 and 4, if $f(x) = 2x^3$ and $g(x) = 6x - 4$, evaluate the expression.

3. a. $f(g(-3))$ b. $f \circ g(-3)$
4. a. $f \circ g(5)$ b. $f(g(5))$
5. Suppose $f(x) = x + 1$ and $g(x) = x - 3$. Is composition of f and g commutative? Justify your answer.

In 6 and 7, let $f(a) = a^2 + a + 1$ and $g(a) = -4a$.

6. Evaluate each expression.
   a. $g(f(-3))$ b. $f(g(-3))$ c. $g(f(3))$ d. $f(g(3))$
7. Find an expression in standard form for each.
   a. $g(f(a))$ b. $f(g(a))$
8. Use a CAS to define the functions f and g in Example 2. Evaluate each expression and compare your results with the answers in the example.
   a. $f(g(x))$ b. $g(f(x))$



9. Refer to Example 3.

a. What is the domain of $f \circ s$?

b. Is the domain of $f \circ s$ the same as the domain of $s \circ f$?

APPLYING THE MATHEMATICS

In 10 and 11, let $g(x) = \frac{1}{x}$ and $h(x) = x^4 - 16$.

10. Find the domain of $g \circ h$.

11. Find the domain of $h \circ g$.

12. Suppose a state has a sales tax of 5% and you buy something at a discount of 20%. Let P be the original price of what you have bought.

a. Write a formula for $f(P)$, the price after the discount.

b. Write a formula for $g(P)$, the price after tax.

c. Which gives you a better final price, discount first or tax first? Explain why.

13. Jarrod earns some extra cash by chauffeuring on the weekends. He rents a limousine from Executive Rentals. According to their rental agreement, Jarrod pays Executive a $300 rental fee plus 40% of his remaining proceeds. Luxury Rental makes Jarrod a different offer. They would charge him $300, but they tell him he will pay them 40% of all collected proceeds *before* deducting the $300 fee. Use function composition to show which offer Jarrod should take.

14. Let $g(x) = x^{\frac{2}{3}}$, where x is a real number. The function g can be rewritten as the composite of two functions m and h. Define m and h so that $h(m(x)) = m(h(x)) = x^{\frac{2}{3}}$.

15. Composite functions can be used to describe relationships between functions of variation. Suppose w varies inversely as z, and z varies directly as the fourth power of x.

a. Give an equation for w in terms of z. Use your equation to describe a function $f: z \rightarrow w$.

b. Give an equation for z in terms of x. Use your equation to describe a function $g: x \rightarrow z$.

c. Give an equation for w in terms of x. Use your equation to describe a function $h: x \rightarrow w$.

d. Use words to describe how w varies with x.

e. How are f, g, and h related?

16. Let $f(x) = x^2$. What is $f(f(f(x)))$?

17. Consider $t(x) = \frac{1}{x^2}$.

a. Simplify $t(t(x))$.

b. When is $t(t(x))$ undefined?

REVIEW

18. Nancy throws a ball upward at a velocity of 18 $\frac{\text{m}}{\text{sec}}$ from the top of a cliff 35 m above sea level. **(Lesson 6-4)**

a. Write an equation to describe the height h of the ball after t seconds.

b. What is the maximum height of the ball?

19. Find the inverse of the matrix $\begin{bmatrix} a & -7 \\ 0 & 18 \end{bmatrix}$, $a \neq 0$. **(Lesson 5-5)**

20. Consider the transformation $T: (x, y) \rightarrow (y, x)$. **(Lesson 4-6)**

a. Let $A = (-4, 0)$, $B = (-1, 5)$, and $C = (2, 5)$. Graph the image of $\triangle ABC$ under T.

b. What transformation does T represent?

21. **True or False** The line with the equation $\pi = x$ is the graph of a function. **(Lesson 1-4)**

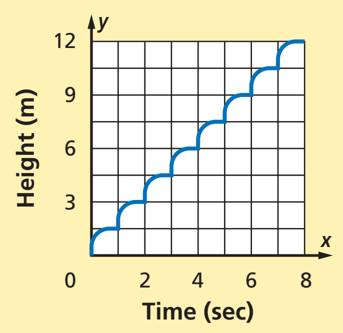
22. The graph at the right shows the height of a flag on a 12-meter pole as a function of time. **(Lesson 1-4)**

a. Describe what is happening to the flag.

b. Why are there some horizontal segments on the graph?

c. What is the domain of the function?

d. What is its range?



23. a. What number is the additive inverse of $-\frac{13}{4}$?

b. What number is the multiplicative inverse of $-\frac{13}{4}$?

(Previous Course)

EXPLORATION

24. Let $f(x) = \frac{1}{1-x}$.

a. Find $f(f(x))$, $f(f(f(x)))$, and $f(f(f(f(x))))$.

b. What is the relationship between $f(x)$ and $f(f(f(f(x))))$?

c. Use your answers to Parts a and b to find $f(f(f(f(f(f(f(f(x))))))))$.

QY ANSWERS

1. $r(d(50{,}000)) = 42{,}000$; $d(r(50{,}000)) = 42{,}240$

2. a. $d \circ r: x \longrightarrow d(r(x))$

b. $d \circ r(x) = d(r(x))$