Lesson 2-9 Combined and Joint Variation

Vocabulary

combined variation

joint variation

BIG IDEA The same methods used to solve variation problems involving two variables can be applied to variation problems involving more than two variables.

Combined Variation

At the beginning of this chapter, you read about how adjusting the number of teeth on the gears of a bicycle changes its speed. The speed S of a bicycle varies directly with the number of revolutions per minute (rpm) R that you turn the pedals and with the number F of teeth on the front gear. The speed also varies inversely with the number B of teeth on the back gear. This situation is modeled by the equation

$$S = \frac{kRF}{B}.$$

This equation is read "S varies directly as R and F and inversely as B." When both direct and inverse variations occur together in a situation, we say the situation is one of **combined variation**.

You saw another example of combined variation in Lesson 2-8, where the maximum weight M of a board varied directly with its width w and the square of its thickness t, and inversely with the distance d between its supports. This relationship was modeled by the equation

$$M = \frac{kwt^2}{d}.$$

STOP QY1

A combined-variation equation has two or more independent variables, and the independent variables can have any positive exponent. To find k in a combined-variation model, use the same strategy as in a variation problem with one independent variable:

- Find one instance that relates all the variables simultaneously.
- Substitute known values into the general variation equation.
- Solve for k.

Mental Math

Jeff is experimenting with a balance scale. He finds that 8 erasers balance 1 apple.

a. His calculator weighs 2.5 times as much as an apple. How many erasers does he need to balance his calculator?

b. A pair of scissors weighs the same as 2 erasers. How many pairs of scissors will balance the calculator?

c. There are two pairs of scissors on one side of the scale and the calculator on the other. How many erasers should he add to the pan with the scissors to balance the calculator?

QY1

Write an equation that represents this statement: y varies directly as the square of x and inversely as z.

Example 1

Mario is pedaling a bike at 80 revolutions per minute using a front gear with 35 teeth and a back gear with 15 teeth. At these settings, he is traveling 14.5 miles per hour. Describe how his speed would change if he increased his pedaling to 100 rpm.

Solution Use the combined variation equation $S = \frac{kRF}{B}$. Substitute $S = 14.5$, $R = 80$, $F = 35$, and $B = 15$ and solve for k.

$$14.5 = \frac{k(80)(35)}{15}$$

$$14.5 \approx k(187)$$

$$0.078 \approx k$$

So the variation formula for this situation is $S = \frac{0.078RF}{B}$.

To find Mario's new speed, substitute $R = 100$, $F = 35$, and $B = 15$ into your formula and solve for S.

$$S = \frac{0.078(100)(35)}{15} \approx 18$$

This means that by increasing his rpm to 100, Mario increases his speed by about 3.5 mph (from 14.5 mph to 18 mph).

Another Example of Combined Variation

Photographers are always looking for ways to make their pictures sharper. One way is to focus the lens at the *hyperfocal distance*. Focusing the lens at this distance will produce a photograph with the maximum number of objects in focus.

Photographers often use the hyperfocal distance when taking pictures of landscapes. The two images at the right were shot with the same camera at the same settings, except the one on the bottom was taken with the lens focused at the hyperfocal distance, giving the extra degree of sharpness.

Example 2 explores how the hyperfocal length H can be calculated using the focal length L of the camera lens in millimeters and the f-stop, or aperture, f. The *aperture* is a setting that tells you how wide the lens opening is on a camera. It is the ratio of the focal length to the diameter of the lens opening and so has no units.

Example 2

The hyperfocal length *H* in meters varies directly with the square of the focal length *L* in millimeters, and inversely with the selected aperture *f*.

a. **Write a general variation equation to model this situation.**

b. **Find the value and unit of *k* when the hyperfocal length *H* is 10.42 m when using a 50 mm lens (*L*) and the aperture *f* is set at 8.**

c. **Write a variation formula for the situation using your answer to Part b.**

d. **Find the hyperfocal length needed if you want to shoot with a 300 mm lens at an aperture setting of 2.8. Include the units in your calculations.**

Solution

a. Because the hyperfocal length varies directly with the square of the focal length of the lens, L^2 will be in the numerator of the expression on the right side of the formula. Because the hyperfocal length varies inversely with the aperture setting, *f* will be in the denominator of the expression.

A general equation is $H = \frac{kL^2}{f}$.

b. Use your formula from Part a with: $H = 10.42$ m, $L = 50$ mm, and $f = 8$. Include units when making substitutions. Here we show how to do it by hand and with a CAS.

By hand:

$10.42 \text{ m} = \frac{k(50 \text{ mm})^2}{8}$ Substitute the given values for *H*, *L*, and *f*.

$10.42 \text{ m} = \frac{2500 \text{ mm}^2 \cdot k}{8}$ Square 50 mm.

$83.36 \text{ m} = 2500 \text{ mm}^2 \cdot k$ Multiply both sides by 8.

$0.033 \frac{\text{m}}{\text{mm}^2} \approx k$ Divide both sides by 2500 mm^2.

With a CAS:

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solve(10.42=k×50^2/8,k)
                {k=0.033344}
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c. Substitute *k* into the equation from Part a.

$$H = \frac{0.033\, L^2}{f}$$

d. Use the formula from Part c to calculate *H* when $L = 300$ mm and $f = 2.8$. Include the units when you substitute.

$$H = \frac{0.033 \frac{\text{m}}{\text{mm}^2}(300 \text{ mm})^2}{2.8}$$

$$H = \frac{0.033 \frac{\text{m}}{\text{mm}^2}(90{,}000 \text{ mm}^2)}{2.8}$$

$$H \approx 1060.7 \text{ m}$$

The hyperfocal length is about 1,060 meters.

Joint Variation

Sometimes one quantity varies directly as powers of two or more independent variables, but not inversely as any variable. This is called **joint variation**. The simplest joint-variation equation is

$$y = kxz,$$

where k is the constant of variation. The equation is read "y varies jointly as x and z" or "y varies directly as the product of x and z." Guided example 3 explores a joint-variation situation in geometry.

GUIDED

Example 3

The volume of a solid with a circular base varies jointly as the height of the solid and the square of the radius of the base.

a. Write a general equation to model this situation.

b. If the volume of the solid is approximately 75.4 cubic centimeters when the radius is 2 centimeters and the height is 6 centimeters, find the value of *k*.

c. The value of *k* is approximately equal to what famous mathematical value?

d. Write a variation formula using your answer to Part b.

e. What well-known kind of solid figure could this be?

Solution

a. Let V be the volume of the solid, h be the height of the solid, and r be the radius of the base.

A general equation is $V = k\underline{\ ?\ }$.

b. Use your formula from Part a with $V = 75.4$, $h = 6$, and $r = 2$ to solve for k.

$75.4 = k\ \underline{\ ?\ }\ \underline{\ ?\ }^2$

$75.4 = k\ \underline{\ ?\ }$

$\underline{\ ?\ } \approx k$

c. The value of k is approximately equal to $\underline{\ ?\ }$.

d. Substitute k into your equation from Part a. So, $V = \underline{\ ?\ }$.

e. Based on the formula in Part d, this solid could be a(n) $\underline{\ ?\ }$.

Many other geometry formulas can be interpreted as direct- or joint-variation equations.

STOP QY2

▶ QY2

Translate this statement into an equation: The area of a triangle varies jointly with the height and the base of the triangle. What is the constant of variation for this formula?

As in combined variation, a joint-variation equation can have more than two independent variables, and the independent variables can have any positive exponent.

Questions

COVERING THE IDEAS

1. a. What is combined variation?
   b. How is joint variation different from combined variation?
2. Translate into a single formula: M varies directly as t and r^2 and inversely as d.
3. Refer to Example 1. Suppose Mario slowed his pedaling to 75 rpm. What would his speed be?
4. In Example 2, assume that a photographer calculated a hyperfocal length of 6.54 meters. If he used a lens with a focal length of 28 millimeters, at what aperture was the lens set?
5. Sonia calculated the volume of a cylindrical can of cat food with a radius of 1.5 inches and a height of 1.2 inches to be about 9.2 cubic inches.
   a. Use the equation $V = kr^2h$ to calculate the constant of variation k that she used.
   b. Did Sonia use the correct formula? How do you know?
6. Refer to Example 3. If the volume of the solid is 25.13 cm^3 with a radius of 2 cm, what is its height?

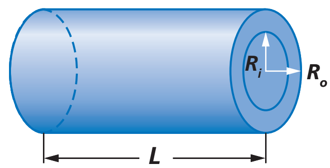
APPLYING THE MATHEMATICS

7. The formula $F = ma$ gives the force F on an object with mass m and acceleration a.
   a. Rewrite the formula in words using the language of variation.
   b. What is the constant of variation?
8. The volume of a solid with a circular base varies jointly with the square of the base radius and the height. When the volume is 83.78 cm^3, the height is 5 cm and the base radius is 4 cm.
   a. Write a general equation to model this situation and find k.
   b. k is a multiple of π. Write k in terms of π.
   c. Write a variation formula for the volume of this solid in terms of π.
   d. What kind of well-known solid is this?

9. The volume of a certain solid varies jointly as its height, width, and length.
   a. Write a general equation to model this situation.
   b. **Fill in the Blank** The answer to Part a suggests that this is a ___?___ solid.
   c. Based on your answer to Part b, find the value of k.

10. Suppose y varies directly as x and inversely as z. Describe how y changes when x and z are each tripled. Explain your answer.

11. The cost C of polyvinyl chloride (PVC) piping in dollars varies jointly as the length L of the pipe and the difference between the squares of its outer and inner radii, $R_o^2 - R_i^2$. Suppose that a foot of PVC piping with an outer radius of 0.25 foot and an inner radius of 0.20 foot costs $3.72.



   a. Using the given variables, write a joint variation equation.
   b. Find the constant of variation.
   c. Rewrite the variation equation using the constant from Part b.
   d. Find the cost of 10 feet of PVC piping with an outer radius of 0.5 foot and in inner radius of 0.48 foot.
   e. Determine the unit of the constant of variation.

REVIEW

12. The Ideal Gas Law in chemistry relates the pressure P (in atmospheres) exerted by a gas to the temperature T (in Kelvins) of the gas and volume V (in liters) of its container. Chantel obtained the following data using a 5-liter container in the lab. (**Lessons 2-7, 2-8**)

T (K)	235	260	285	305	500
P (atm)	0.7285	0.8060	0.8835	0.9455	1.55

   a. Graph the data points.
   b. How does P vary with T?
   c. With a temperature of 350 Kelvins, Chantel manipulated the volume of the container and obtained the following data. Graph these points on a different set of axes.

V (L)	1	2	3	4	5	6
P (atm)	5.425	2.713	1.808	1.356	1.085	0.603

   d. How does P vary with V?
   e. Write an equation that relates P, T, and V. You do not need to find the constant of variation.

13. Let $g: x \rightarrow 3x^2$. (**Lesson 2-5**)

a. Graph g over the domain $-2.5 \le x \le 2.5$.

b. What is the name of the shape of this graph?

c. Find the average rate of change of g between $x = 1$ and $x = 2$.

d. Would the answer to Part c be the same for any two values of x that differ by 1? Explain your answer.

14. If y varies directly as w^3, and $y = 25$ when $w = 5$, find the value of y when $w = 2$. (**Lesson 2-1**)

15. One general equation for a combined variation is $y = k\frac{xz}{w}$. Solve for k in terms of the other variables. (**Lesson 1-7**)

16. When Clara tried to solve the equation $\frac{1}{5}x + \frac{1}{7}x + 2 = 5$, her first step led to $7x + 5x + 70 = 175$. (**Lesson 1-6**)

a. Explain what Clara did.

b. Finish solving the equation.

17. Given the function $f: x \rightarrow 4x^3 - 2x + 1$, find $f(\pi)$. (**Lesson 1-3**)

EXPLORATION

18. There are many instances of mathematics on TV shows. Watch a show in which mathematics plays a role and record any mathematics used on the show. Note whether the mathematics was used accurately and try to explain any mistakes.

QY ANSWERS

1. $y = \frac{kx^2}{z}$

2. $A = kbh$. In this case, $k = \frac{1}{2}$.